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# RESULTS ON LOCATION AND SCALE PARAMETER ESTIMATION WITH APPLICATION TO THE EXTREME-VALUE DISTRIBUTION

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AVIATION, INC.
CANOGA PARK, CALIFORNIA

Contract No. AF 33(615)-2818
Project No. 7071





**OFFICE OF AEROSPACE RESEARCH**

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# RESULTS ON LOCATION AND SCALE PARAMETER ESTIMATION WITH APPLICATION TO THE EXTREME-VALUE DISTRIBUTION


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## FOREWORD

The major portion of the work summarized in this interim technical report was sponsored by the Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio under Contract AF33(615)-2818. Work on this contract, technically monitored by Dr. H. L. Harter, is documented under Project 7071, Research in Applied Mathematics. The tables for obtaining the best linear invariant estimates of parameters of the extreme-value distribution which appear in Appendix C were calculated under the auspices of the Reliability Subdivision of the Liquid Rocket Division at Rocketdyne, with the support of Contract NAS 8-19 (J2 Engine System Development). The initial work on the tables appearing in Appendix D was also performed under the NAS contract.

## ABSTRACT

This report gives results concerning estimation of location and scale parameters. Most of the work pertains to the first extreme-value distribution of smallest values, the distribution of the natural logarithms of failure times having the two-parameter Weibull distribution. Experimental designs are derived, under the assumption that log failure times are polynomial functions of the reciprocal of stress level and have the extreme-value distribution, for over-stress life tests. These designs yield least-squares curves with minimum variance at a specified (nominal) stress level below the levels at which the life test is conducted. An estimate of the extreme-value location parameter $u$ associated with the nominal stress level and the relationship between $u$ and stress level can be obtained from the least-squares curve. Other extreme-value results apply to a life test conducted at a single fixed stress level.

Interval estimators (confidence bounds) are derived for the extreme-value scale parameter $b$ and for any location parameter of the form $x_R = u + b\log\log(1/R)$ where $R$ is specified (the $100(1-R)$ percent point of the distribution). The bounds for $b$ and $x_R$ are, respectively, based on two and three of the first $m$ extreme-value order statistics, $X_1, \le X_2, \ldots, \le X_m$, and are of the form $k_b(X_q - X_p)$ and $X_\nu + k_R(X_q - X_p)$. For each combination of values of sample size $n$, censoring number $m$, confidence level $1-\alpha$, and survival proportion $R$, an optimum choice of $p$, $q$, and $\nu$ has been made. The criterion used in the selection is that the expected squared deviation of the bound from the parameter should be minimized. For samples up to size 22, values of the constant $k_b$ corresponding to the optimum combination of $p$ and $q$ are presented. Tables of values of $k_R$ are also in preparation.

For $2 \leq n \leq 25$, $2 \leq m \leq n$, tables are given for estimating $b$ and $x_R$, where the estimator is best among linear estimators with expected loss invariant under translations. These best linear invariant (BLI) estimators have uniformly smaller expected loss than the Gauss-Markov best linear unbiased (BLU) estimators and are simple linear functions of the BLU estimators.

Expressions are derived for Cramér-Rao bounds for invariant estimators of general location and scale parameters. These bounds are applied to the extreme-value distribution, and compared with the expected losses of BLI and BLU estimators of $u$, $b$, and $x_R$, for certain values of $R$, for $2 \leq n \leq 25$, $2 \leq m \leq n$.

The one problem considered which concerns the Gaussian rather than the extreme-value distribution is the outlier problem for a size-three sample. One approach to the problem involves a simplified Bayesian analysis. This leads to recommendations for estimators of the Gaussian location parameter which depend only on the observations and a prior probability that the largest (or smallest) observed value is an outlier.

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EXTREME-VALUE DISTRIBUTION

EVALUATION OF CONFIDENCE BOUNDS FOR DISTRIBUTION PERCENTAGE POINTS AND FOR THE SCALE PARAMETER

Appendix B gives an analysis of the problem of obtaining confidence bounds on reliability parameters when the natural logarithms of failure times have the Type I extreme-value distribution of smallest values, and $(n-m)/n$ of the size $n$ sample of ordered observations is censored from above. Exact lower confidence bounds on log reliable life $x_R$ (the $100(1-R)$ percent point, where $R$ is a specified survival proportion) are derived. These have smallest mean squared deviation from $x_R$ among bounds of a specified form based on three order statistics only. The computer program and various subroutines for generating the tables for obtaining these bounds have been written. The subroutines, which perform the numerical integration for calculating distribution percentage points, calculate the risk (the expected squared deviation) of the bounds, read from tape the expected values, variances, and covariances of the reduced extreme-value order statistics, generate first guesses for percentage points, sort and order the risks for the various combinations of order statistics, etc., are all working as desired. The calculations based on the program will be made on the Wright-Patterson direct-coupled system due to the fact that a great deal of computer time will be required. This is true even though an approximation is being made in order to decrease substantially the number of distribution percentage points which will be calculated. If no such approximation is made, then for each combination of confidence level $\gamma$ and specified $R$ there are 20,615 percentiles which must be calculated, if only sample sizes 2, 3, ..., 20 are considered. Since several seconds, at

least, appear to be required for calculating each percentage point, it might take well over 2 days of computer time to determine the necessary values if only 2 values each of $\gamma$ and $R$ are considered.

It has therefore been decided to choose the three order statistics on which to base the bounds before calculating any percentage points. For each combination of sample size $n$, censoring number $m$, and specified survival proportion $R$, the estimator of $x_R$ with smallest risk among those based on a linear combination, with risk invariant under translations, of one order statistic and the difference of two order statistics will be selected. The three order statistics specified by this estimation rule will then provide the basis for bounds at all confidence levels for each corresponding combination of $m$, $n$, and $R$. Indications are that very little increase in risk for the bounds will result from this procedure; and the number of computations will be decreased by a factor of over 100.

There is no apparent method of comparing the bounds directly with bounds, such as those obtained by Monte Carlo procedures by Johns and Lieberman [3], based on all of the first $m$ order statistics. It is possible, however, to compare the risks for the three-order-statistic estimators described above with those of the best linear invariant (BLI) estimators, approximations to which form the basis for the bounds obtained by Johns and Lieberman. This has been done for sample sizes 2, 3, ..., 17, and demonstrates that the ratio of the risks of the BLI estimator and the best three-order-statistic estimator of $x_{.90}$ as described above is 0.877 for $n$ as large as 17 and $m$ equal to $n$. Ratios of this type are being determined for certain combinations of $n$, $m$, $R$, and will appear in a future report. Also included will be ratios of risks of the BLI estimator and the best estimator

of the extreme-value scale parameter b based on the two order statistics yielding the bound with smallest risk, for certain combinations of n, m, and $\gamma$.

Table B.I, which provides the means for calculation of the upper confidence bounds on b with smallest mean squared deviation from b among bounds based on two order statistics only, is included in Appendix B. At the same time distribution percentage points were being calculated as a basis for Table B.I, the power function of the tests associated with the bounds were also calculated for three prescribed values of $b/b_o$ (where the hypothesis tested is $H: b \geq b_o$). The calculations showed that the two-order-statistic bound with smallest risk seems to be either uniformly most powerful, locally most powerful, or most powerful for $b/b_o$ close to zero among two-order-statistic bounds. Moreover, when another two-order-statistic bound has greater power than the one with smallest risk for one of the prescribed values of $b/b_o$, the difference between the calculated values of the two is almost negligible.

## CRAMÉR-RAO EFFICIENCIES OF BEST LINEAR INVARIANT ESTIMATORS

The paper included as Appendix D gives the derivation of Cramér-Rao type bounds for invariant estimators of a scale parameter $\theta_2$ and location parameters of the form $\theta_1 + \ell\theta_2$, where $(\theta_1, \theta_2)$ is a location-scale parameter (that is, $f_{\theta_1, \theta_2}(x) = \frac{1}{\theta_2} g\left(\frac{x-\theta_1}{\theta_2}\right)$, for some g). These bounds, which are appropriate for the case in which any proportion of the sample observations may be censored, are shown to be functions of the bounds for unbiased estimators. They are applied specifically to invariant estimators of percentage points of the first extreme-value distribution of smallest values

for the case in which $(n-m)/n$ of a size $n$ sample of ordered observations is censored from above. The Cramér-Rao efficiencies of the best linear invariant estimators are then tabulated (see Table D.III) for the 10 percent point (reliable life for a specified survival proportion $R$ of 0.90) and $2 \leq n \leq 18$. Also tabulated on the basis of previous computations and the derived result (see Table D.I and Table D.II) are the mean squared errors of the best linear invariant and the best linear unbiased estimators, along with their respective Cramér-Rao bounds, for the extreme-value location parameter $u$, for the scale parameter $b$, and for the first and tenth percentiles of the distribution.

This paper was written as an extra task under the contract for the symposium on reliability and life-testing presented by the Society for Industrial and Applied Mathematics (SIAM) in Seattle, November 14-15, 1965. A special issue of the journal of the society was planned to include many of the papers presented at the symposium, and this material was submitted for consideration. The plans for the special issue were apparently dropped, however, so that the paper was accepted for publication in a regular issue of the SIAM Journal, subject to a few revisions in the introduction and an incorporation of the appendices into the body of the paper. Also, Table D.I and Table D.II will not be included. The elimination of these two tables provides part of the motivation for including this paper in this report, since, for example, there is no source other than Table D.II from which one may directly obtain the variances of the best linear unbiased estimators of the extreme-value parameters for $n = 21, 22, \ldots, 25$. (They may be obtained indirectly from Table D.I or from Table C.I in Appendix C). The values of the two types of bounds, too, are of interest for comparison with each other and with the risks of other types of estimators.

## TABLES FOR OBTAINING BEST LINEAR INVARIANT ESTIMATES

The entries for Table C.I, which appears in Appendix C, are based on computations made in part at the Health Sciences Computing Facility at the University of California at Los Angeles and in part at Rocketdyne, under the auspices of the Liquid Rocket Division with the support of Contract NAS 8-19. The values computed at that time (weights for obtaining best linear unbiased estimates of extreme-value parameters) were modified according to theory recently derived to obtain the weights for the best linear invariant (BLI) estimates. These weights for the BLI estimates were then published as part of a Rocketdyne research report under the contract. The report is included here because it has elicited considerable interest and appears to warrant a wide distribution. The paper has been submitted for publication but may present something of a problem in this respect because of the length of the tables.

## DESIGN OF OVER-STRESS LIFE-TEST EXPERIMENTS

Appendix A gives the derivation of results specifying designs with certain optimum properties for life-test experiments under particular assumptions concerning the failure times of the population of items under consideration. A two-parameter Weibull model is assumed for the probability density function of the failure times; and further it is supposed that the logarithm of failure time is a polynomial function of the reciprocal of stress level. Finally, it is assumed that the level of stress to which the population of items being considered is subjected during normal usage (the nominal stress level), is not sufficient to produce a failure of an item within any reasonable amount of time after the application of the stress. Thus it is supposed that the life test,

which will allow one to estimate certain properties of the distribution of the failure times at the nominal stress level, is conducted at levels above nominal stress.

The designs derived under these assumptions specify where (at what stress levels) within a given interval of stress the life tests should be conducted and what proportion of the total number of items should be assigned to each testing level when all of the items are tested until failure and when least-squares estimation procedures are to be applied. Certain properties are demonstrated for the least-squares curve at the various testing levels and at the nominal stress level when a design of the form derived is used.

It should be noted that the assumption originally proposed for this task concerning a change of level of stress applied to an item during a life test was not incorporated in this work. The reason for this was the fact that the following became apparent after some analysis. For testing at various stress levels where any failure-time distribution parameter depends upon stress level, it is not possible to estimate such a parameter efficiently at any level unless the relationship between the parameter and the stress level can also be estimated. When stress level is changed during testing, it is not clear how one should determine or define the actual stress which has been applied to produce the failure. In other words, the testing level is not actually defined for this situation so that it is not possible to estimate the relationship between stress level and failure-time parameters.

## THE OUTLIER PROBLEM FOR SAMPLES OF SIZE THREE

### DEFINITION OF THE PROBLEM

Suppose three observations on a normally distributed random variable are made, and the largest (or smallest) is suspected of being an outlier. The chemist's approach to the problem of estimating the mean $\mu$ of the distribution has traditionally been to discard the observed value of the possible outlier and average the other two values. Use of a Winsorized estimator of $\mu$, often recommended when outlying values are suspect, involves changing the observed value of the outlier to the value of the second ordered observation, and then averaging all three values. Sometimes the median of the three observations is used. Clearly, if the suspected value is not an outlier, or if the amount by which it is displaced is very small compared with the distribution standard deviation then the arithmetic mean of all three observed values is best for estimating $\mu$. What if, however, the amount of displacement is moderate or large or there is a fairly high probability that the observation in question is an outlier? The following analysis gives some insight into this problem.

### COMPARISON OF ESTIMATORS FOR THE MEAN

Assume that $Y_1 \leq Y_2 < Y_3$, and that either $\theta_o$: $Y_1$, $Y_2$, and $Y_3$ are order statistics from $N(\mu,\sigma)$, or $\theta_\Delta$: $Y_1$ and $Y_2$ are order statistics from $N(\mu,\sigma)$ and $Y_3$ is a size 1 sample from $N(\mu + \Delta,\sigma)$. Because of the symmetry of the distribution, the case in which $Y_1$ is the observation which is suspect can also be handled.) The risk (expected loss, where loss is squared error divided by $\sigma^2$) can then be calculated for any estimator under the two alternatives $\theta_o$ and $\theta_\Delta$. All values given below are correct to within a unit in the final decimal place shown. Values which are exact are given in fractional form. The moments of the normal

order statistics used to calculate these values appear in [4].

Consider $\hat{\mu}_0 = \frac{1}{3}(Y_1 + Y_2 + Y_3)$

$$\hat{\mu}_1 = \frac{1}{2}(Y_1 + Y_2)$$

$$\hat{\mu}_2 = \frac{1}{3}(Y_1 + 2Y_2)$$

and $\hat{\mu}_3 = Y_2$ .

If $R_k(\mu,\hat{\mu}_j)$ is the risk for estimator $\hat{\mu}_j (j=0,1,2,3)$ when $\theta_k (k=0 \text{ or } \Delta)$ is true, then

$$R_o(\mu,\hat{\mu}_o) = \frac{1}{3} \qquad R_\Delta(\mu,\hat{\mu}_o) = \frac{1}{3} + \Delta^2/9\sigma^2$$

$$R_o(\mu,\hat{\mu}_1) = .569 \qquad R_\Delta(\mu,\hat{\mu}_1) = \frac{1}{2}$$

$$R_o(\mu,\hat{\mu}_2) = .464 \qquad R_\Delta(\mu,\hat{\mu}_2) = .556$$

$$R_o(\mu \hat{\mu}_3) = .449 \qquad R_\Delta(\mu \hat{\mu}_3) = 1$$

Of the three estimators $\hat{\mu}_1, \hat{\mu}_2$, and $\hat{\mu}_3$, $\hat{\mu}_2$ has minimax risk (minimizes the maximum risk for all $\Delta$, $\mu$, and $\sigma$). It is not the true minimax estimator, however. Assume $\sigma$ is not known and consider all estimators which are linear functions of $Y_1$ and $Y_2$ and which have expected loss indep' ient of $\mu$ and $\sigma$. These linear invariant estimators will be of the form $\hat{\mu}(a_1) = a_1 Y_1 + (1-a_1)Y_2$ [5].

When $\theta_o$ is true, the best linear unbiased estimators of $\mu$ and $\sigma$ based on $Y_1$ and $Y_2$ arr simply $\mu_o^* = Y_2$ and $\sigma_o^* = 1.1816 (Y_2 - Y_1)$

with the variances and covariance of these estimators respectively given by $V(\mu_o^*) = .4487\sigma^2 \equiv A_o\sigma^2, V(\sigma_o^*) \equiv .6378\sigma^2 \equiv C_o\sigma^2, C(\mu_o^*,\sigma_o^*) = .2044\sigma^2 \equiv B_o\sigma^2$. Thus, by the results in [ ], when $\theta_o$ is true the best linear invariant estimator based on $Y_1$ and $Y_2$ is $\tilde{\mu}_o = \mu_o^* - [B_o/(1 + C_o)]\sigma_o^*$ $= .1475Y_1 + .8525Y_2$. When $\theta_\Delta$ is true $\mu_\Delta^* = \hat{\mu}_1 = \frac{1}{2}(Y_1 + Y_2)$ and $\sigma_\Delta^* = -.8862Y_1 + .8862Y_2$ are, respectively, the best linear unbiased estimators of $\mu$ and $\sigma$. Since the covariance $B_\Delta\sigma^2$ of $\mu_\Delta^*$ and $\sigma_\Delta^*$ is equal to zero, the best linear invariant estimator $\tilde{\mu}_\Delta$ based on $Y_1$ and $Y_2$ for $\theta_\Delta$ true is equal to $\mu_\Delta^*$. The risk $R_k(\mu,\tilde{\mu}_j)$ for $\tilde{\mu}_j$ when $\theta_k (j,k=0,\Delta)$ is true is given as follows

$$R_o(\mu,\tilde{\mu}_o) = .423 \qquad R_\Delta(\mu,\tilde{\mu}_o) = .748$$

$$R_o(\mu,\tilde{\mu}_\Delta) = .569 \qquad R_\Delta(\mu,\tilde{\mu}_\Delta) = \tfrac{1}{2} \; .$$

Now, since $\tilde{\mu}_o$ is not equal to $\tilde{\mu}_\Delta$, which would imply $R_o(\mu,\tilde{\mu}_o)$ $= R_o(\mu,\tilde{\mu}_\Delta)$ and $R_\Delta(\mu,\tilde{\mu}_o) = R_\Delta(\mu,\tilde{\mu}_\Delta)$, there is no guarantee that $\tilde{\mu}_\Delta$ has minimax risk among all linear functions of the form $\hat{\mu}(a_1)$. In fact, $\tilde{\mu}_\Delta = \hat{\mu}_1$ has larger maximum risk than $\hat{\mu}_2$, as was noted earlier. Define $\hat{\mu}_4$ to be the minimax-risk linear invariant estimator, that is, the one such that $\max[R_o(\mu,\hat{\mu}_4), R_\Delta(\mu,\hat{\mu}_4)]$ is the minimum of $\max[R_o(\mu,\hat{\mu}(a_1)), R_\Delta(\mu,\hat{\mu}(a_1))]$, $-\infty < a_1 < \infty$. The maximum risk of $\hat{\mu}_4$ can be no smaller than $\max[R_o(\mu,\tilde{\mu}_o), R_\Delta(\mu\ \tilde{\mu}_\Delta)] = R_\Delta(\mu,\hat{\mu}(1/2)) = \frac{1}{2}$. Furthermore $R_o(\mu,\hat{\mu}(a_1))$ and $R_\Delta(\mu,\hat{\mu}(a_1))$ are continuous quadratic

functions of $a_1$, each with a unique minimum. Therefore, if the value $\bar{a}_1$ of $a_1$ defined by

$$R_o(\mu,\hat{\mu}(\bar{a}_1)) = R_\Delta(\mu,\hat{\mu}(\bar{a}_1)) \tag{1}$$

is such that $.1475 < \bar{a}_1 < \frac{1}{2}$ (note that $\tilde{\mu}_o = \hat{\mu}(.1475)$ and $\tilde{\mu}_\Delta = \hat{\mu}(1/2)$), then $\hat{\mu}(\bar{a}_1)$ is the minimax estimator $\hat{\mu}_4$. Solution of equation (1) gives $\bar{a}_1 = .42265$. Therefore, $\hat{\mu}(\bar{a}_1) = \hat{\mu}_4 = .42265Y_1 + .57735Y_2$, with $R_o(\mu,\hat{\mu}_4) = R_\Delta(\mu,\hat{\mu}_4) = .512$. Moreover, $\hat{\mu}_4$ can be expressed as $.5(Y_1 + Y_2) + .0830[.8862(Y_2 - Y_1)] = \mu_\Delta^* + .0830\sigma_\Delta^*$, with $\sigma_\Delta^*$ equivalent to the unique best unbiased estimator of $\sigma$ for sample size 2. Therefore, for $\theta_\Delta$ true, distribution percentage points of $\frac{\hat{\mu}_4-\mu}{\sigma^*}$ can be calculated from percentage points of the central $t$ distribution.

In a real situation one might not want to use the minimax estimator, particularly if $|\Delta/\sigma|$ is apt to be small. A simplified Bayesian analysis may be employed to gain insight into how large $|\Delta/\sigma|$ must be before one would prefer another estimator to $\hat{\mu}_o$. Let $\gamma$ be the prior probability of situation $\theta_\Delta$, with $|\Delta/\sigma|$ fixed, and $1-\gamma$ the probability of $\theta_o$. For comparative purposes the estimators $\hat{\mu}_o,\hat{\mu}_1,\hat{\mu}_2,\hat{\mu}_3$, and $\hat{\mu}_4$ will be used.

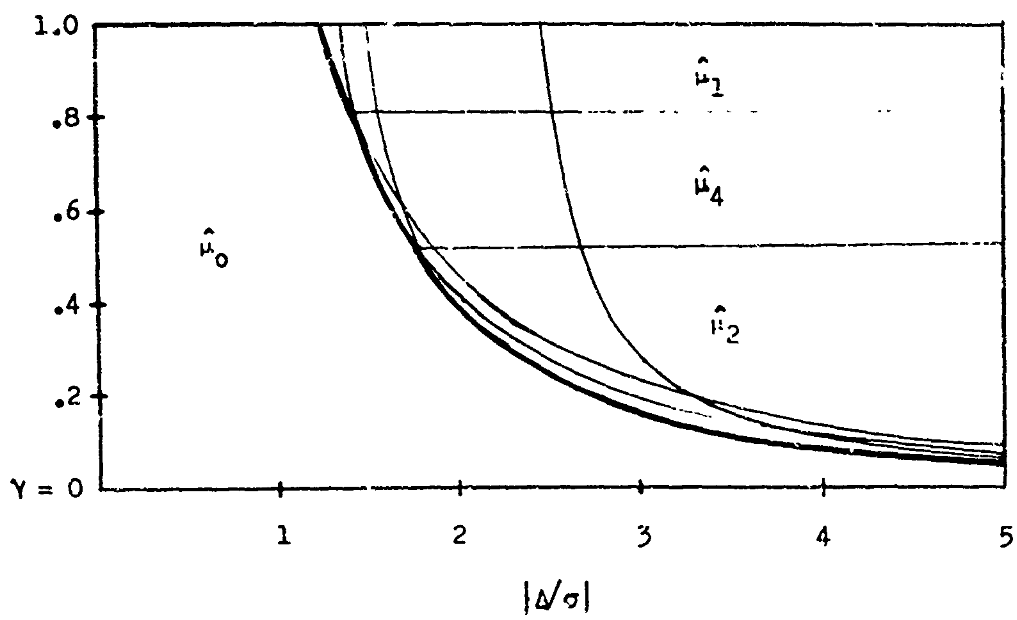
The risk using the estimator $\hat{\mu}_o$ is $1/3$ with probability $1-\gamma$ and $1/3 + \Delta^2/9\sigma^2$ with probability $\gamma$. The risk function for $\hat{\mu}_o$ can thus

be expressed as a function of $\gamma$ and $\Delta$, $R(\mu,\hat{\mu}_o, \gamma, \Delta) = 1/3+\gamma(\Delta^2/9\sigma^2)$.
For $\hat{\mu}_i$, $i=1,2,3,4$, the risk function $R(\mu,\hat{\mu}_i,\gamma,\Delta)$ is independent of $\Delta$ and can be expressed in terms of $\gamma$ alone:

| $i$ | $R(\mu,\hat{\mu}_i, \gamma, \Delta)$ |
|---|---|
| 0 | $1/3 + \gamma\Delta^2/9\sigma^2$ |
| 1 | $.569 - .070\gamma$ |
| 2 | $.464 + .092\gamma$ |
| 3 | $.449 + .551\gamma$ |
| 4 | $.512$ |

The last four of these expressions indicate the following: Of the estimators $\hat{\mu}_1,\hat{\mu}_2,\hat{\mu}_3,\hat{\mu}_4$, $\hat{\mu}_1 = \frac{1}{2}(Y_1 + Y_2)$ is to be preferred for $\gamma > .82$, for $.52 < \gamma \leq .82$, $\hat{\mu}_4$, the minimax estimator, and for $.03 < \gamma \leq .52$, $\mu_2$, the Winsorized estimator $\frac{1}{3}(Y_1 + 2Y_2)$, are preferable. The median $\hat{\mu}_3$ is preferable to the other three only for $\gamma < .03$.

Figure 1




In practical situations it is likely that $\gamma$ would be of small or moderate size and that $\Delta/\sigma$ would be positive and moderate or large. (If in fact $|\Delta/\sigma|$ were small one would not be apt to suspect the presence of outliers.) Thus, any of $\hat{\mu}_0$, $\hat{\mu}_2$, or $\hat{\mu}_4$ might be the preferable estimator. The median would be preferred to the other four estimators only if both $\gamma < .03$ and $|\Delta/\sigma| \geq 6$. It seems clear from this analysis that for each $\gamma$ there is an estimator $\hat{\mu}(a_1(\gamma)) = a_1(\gamma)Y_1 + (1-a_1(\gamma))Y_2$ which minimizes the value of $|\Delta/\sigma|$ required such that a two-order-statistic estimator is preferred to $\hat{\mu}_0$.

The expression for $a_1(\gamma)$ has been found to be given by

$$a_1(\gamma) = \frac{.17301(1-\gamma)+\gamma}{1.17301(1-\gamma)+2\gamma}.$$

Hence,

$$a_1(1) = \tfrac{1}{2},\ \hat{\mu}(a_1(1)) = \tfrac{1}{2}(Y_1+Y_2),$$

$$a_1(.5) = .3697,\ \hat{\mu}(a_1(.5)) = .3697Y_1 + .6303Y_2,$$

and

$$a_1(0) = .1474,\ \hat{\mu}(a_1(0)) = .1474Y_1 + .8526Y_2,$$

where $\hat{\mu}(a_1(1))$ is $\hat{\mu}_\Delta$, the best linear invariant (BLI) estimator when $\theta_\Delta$ is true, and $\hat{\mu}(a_1(0))$ is $\hat{\mu}_0$, the BLI estimator when $\theta_0$ is true. For $\hat{\mu}(a_1(\gamma))$ equal to the minimax estimator, $a_1(\gamma) = .42265$, so that $\gamma$ is equal to .676.

It has also been demonstrated that $a_1(\gamma)$ increases monotonically with $\gamma$ from .1474 for $\gamma = 0$ to .5 for $\gamma = 1$. Thus, the median would never be the two-order statistic estimator recommended in preference to $\hat{\mu}_0$.

## A METHOD OF DEALING WITH THE UNKNOWN PARAMETER $\Delta/\sigma$

A Monte Carlo analysis of a more general outlier problem is discussed in the final portion of Ferguson [2]. The results indicate that for a sample size as small as 3, there are no discernible differences between the values of the power functions of the locally most powerful test of $\Delta/\sigma = 0$ versus $\Delta/\sigma > 0$, based on the coefficient of skewness and two other tests, one of which is based on $R_{10} = (X_n - X_{n-1})/(X_n - X_1)$, with $X_1 \leq X_2 \leq \ldots \leq X_{n-1} \leq X_n$, proposed by Dixon in [1].

Suppose we consider $\left(\frac{1}{R_{10}} - 1\right)^{-1} = \left\{\left[(Y_3 - Y_2)/(Y_3 - Y_1)\right]^{-1} - 1\right\}^{-1}$ $= (Y_3 - Y_2)/(Y_2 - Y_1)$. Note that if $Z_{i,n}$ is the $i^{th}$ order statistic from a size $n$ sample from the standard normal distribution, $\left[(Y_3 - Y_2)/(Y_2 - Y_1)\right](Z_{2,2} - Z_{1,2}) + Z_{2,2} \equiv \Delta/\sigma + Z_{1,1}$ with probability $\gamma$, and $\left[(Y_3 - Y_2)/(Y_2 - Y_1)\right](Z_{2,3} - Z_{1,3}) + Z_{2,3} \equiv Z_{3,3}$, with probability $1 - \gamma$, where $(Y_i - \mu)/\sigma = Z_{i,2}$ $(i = 1,2)$, and $(Y_3 - \mu)/\sigma = \Delta/\sigma + Z_{1,1}$ with probability $\gamma$ and $(Y_i - \mu)/\sigma = Z_{i,3}$ and $(Y_3 - \mu)/\sigma = Z_{3,3}$ with probability $1 - \gamma$. Let $\left[(Y_3 - Y_2)/(Y_2 - Y_1)\right](Z_{2,2} - Z_{1,2}) + Z_{2,2} - Z_{1,1}$ be identically equal to $t(Z_{2,2} - Z_{1,2}) + Z_{2,2} - Z_{1,1} \equiv W(t)$, and let $\overline{\Delta/\sigma} > 0$ be specified. Then $\Pr\left[\Delta/\sigma > \overline{\Delta/\sigma}\right] = \Pr\left[\theta_\Delta \text{ is true}\right] \times \Pr\left[\Delta/\sigma > \overline{\Delta/\sigma} \mid \theta_\Delta \text{ is true}\right]$ or

$$\Pr\left[\Delta/\sigma > \overline{\Delta/\sigma}\right] = \gamma \int_{\overline{\Delta/\sigma}}^{\infty} dF\left[W(t)\right] .$$

The expression on the right-hand side of this equation can be calculated by considering the joint density of $Z_{1,2}$, $Z_{2,2}$ and $Z_{1,1}$. From this, one obtains

$$\gamma \int_{\overline{\Delta/\sigma}}^{\infty} dF[W(t)] = \gamma\left\{1 - \frac{1}{\sqrt{3(2t+1)}}\left\{3\left[\Pr\left(Z < \frac{(2t+1)\overline{\Delta/\sigma}}{\sqrt{6(1+t+t^2)}}\right)\right]^2\left[\Pr\left(Z > \frac{(2t+1)\overline{\Delta/\sigma}}{\sqrt{6(1+t+t^2)}}\right)\right] + \left[\Pr\left(Z < \frac{(2t+1)\overline{\Delta/\sigma}}{\sqrt{6(1+t+t^2)}}\right)\right]^3\right\}\right\},$$

where $Z$ is the standard normal deviate and $t$ is greater than zero.

The expression $\dfrac{2t+1}{\sqrt{6(1+t+t^2)}}$ is equal to $\dfrac{2Y_3 - Y_2 - Y_1}{\sqrt{6[(Y_3-Y_2)^2 + (Y_3-Y_2)(Y_2-Y_1)+(Y_2-Y_1)^2]}}$

and $2t + 1$ equals $\dfrac{2Y_3 - Y_2 - Y_1}{Y_2 - Y_1}$ .

An appropriate value for $\overline{\Delta/\sigma}$ in terms of $\gamma$ alone can be obtained by substituting $a_1(\gamma)$ into the original expression from which the value for $a_1(\gamma)$ is derived. This gives

$$\overline{\Delta/\sigma} = \left[ \frac{2}{\gamma} \frac{(.18853\gamma^3 + .99076\gamma^2 - .20923\gamma + 1.029932)}{(.68391\gamma^2 + 1.9401\gamma + 1.3759)} - \frac{3}{\gamma} \right]^{\frac{1}{2}} .$$

Therefore, for any prior probability $\gamma$ that $\theta_\Delta$ is true, the probability that $\Delta/\sigma$ is sufficiently large so that the risk of a two-order-statistic estimator is smaller than the risk of $\hat{\mu}_0$ can be calculated as a function of $\gamma$ and the observations only. One could then choose some fixed value $P$, $0 < P < 1$ (a good choice might be $P = .5$), and for any known $\gamma$, estimate $\mu$ by $\hat{\mu}_0 = \frac{1}{3}(Y_1 + Y_2 + Y_3)$ if $\Pr\left[\Delta/\sigma > \overline{\Delta/\sigma}\right]$ is less than $P$ and by $\hat{\mu}(a_1(\gamma))$ otherwise.

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## APPENDIX A

## DESIGN OF OVER STRESS LIFE-TEST EXPERIMENTS WHEN FAILURE TIMES HAVE THE TWO-PARAMETER WEIBULL DISTRIBUTION


### SUMMARY

The following paper deals with the situation in which estimation of reliability parameters associated with a failure-time distribution requires that life testing be conducted at stress levels above that which would normally be applied in standard usage (the nominal stress level). This situation may prevail when an inordinate amount of time is required to obtain a failure at the nominal stress level.

A range of stress for testing is prescribed, and a two-parameter Weibull model for failure times (a type I extreme-value distribution of smallest values for log failure time) is assumed. The extreme-value location parameter u, the logarithm of the Weibull scale parameters, is assumed to be a polynomial function of known degree k of the reciprocal of stress level. The Weibull shape parameter b, which is also the extreme-value scale parameter, is assumed to be independent of stress level. The number n of items in the total sample to be life tested is given, and it is assumed that stress level for each life test is sufficiently high so that each item can be tested until failure. The problem considered is that of determining the design for obtaining the least-squares-curve intercept with minimum variance at the nominal stress level. The design to be determined consists of the number and location of stress levels in the prescribed range at which the life tests will be conducted and proportion of total sample to be randomly allocated to each testing level.

The results of Hoel and Levine in [2] concerning determination of designs for minimum-variance polynomial extrapolation cannot be applied directly because the models for which these results are appropriate are those in which the observations are normally distributed, uncorrelated, and have uniform variance. In our model the logarithms of observed failure times have the Type I extreme-value distribution of smallest values, unequal variances, and covariances that are not zero except between stress levels. It is shown that the methods of proof utilized by Hoel and Levine are applicable, however, and rules for determining designs yielding the "approximately" minimum-variance least-squares intercept for the prescribed model are derived.

One of the interesting results in the derivation of the design is the delineation of the point through which the least-squares curve for the observations passes at the $\alpha^{th}$ stress level, $\alpha = 0, 1, \ldots, \ell - 1$, when the number $\ell$ of stress levels at which observations are made is $k + 1$. This point is $\frac{1}{\sigma_\alpha}, X^*\left(\frac{1}{\sigma_\alpha}\right)$, where $X^*\left(\frac{1}{\sigma_\alpha}\right)$ is a weighted average of the ordered observations and where the $i^{th}$ weight consists of the elements in the $i^{th}$ row, $i = 1, 2, \ldots, m_\alpha$, of the covariance matrix of the $m_\alpha$ ordered observations made at the $\alpha^{th}$ level, $\alpha = 0, 1, \ldots, k$. For $m_\alpha \geq 2$, $X^*\left(\frac{1}{\sigma_\alpha}\right)$ can also be expressed as a specific function of the Gauss-Markov least-squares best linear unbiased estimators $u^*\left(\frac{1}{\sigma_\alpha}\right)$ and $b^*_\alpha$ at $\sigma = \sigma_\alpha$ of $u_\alpha$ (the value of $u$ at the $\alpha^{th}$ testing level) and

b, respectively, the variance of $b_\alpha^*$ and the covariance of $u^*\left(\frac{1}{\sigma_\alpha}\right)$ and $b_\alpha^*$, $\alpha = 0, 1, \ldots, k$. This particular result is independent of the distribution of the observations and applies whether or not $m_\alpha$ is equal to $n_\alpha$, where $n_\alpha = p_\alpha n$ and $p_\alpha$ is the proportion of the sample randomly assigned to the $\alpha^{th}$ sample point, $\alpha = 0, 1, \ldots, k$, (that is, there may be censoring). It is shown that there is no linear combination $\bar{X}\left(\frac{1}{\sigma_\alpha}\right)$ of ordered observations at the $\alpha^{th}$ testing level and no point along that ordinate such that the mean squared deviation of $\bar{X}\left(\frac{1}{\sigma_\alpha}\right)$ from the point is less than or equal to the variance of $X^*\left(\frac{1}{\sigma_\alpha}\right)$ for all $u_\alpha$ and b, $\alpha = 0, 1, \ldots, k$. Similar properties are demonstrated for the least-squares intercept at the nominal stress level under the design prescribed by the derived results.

## INTRODUCTION

Often an engineer may want an estimate of a parameter associated with the reliability of a population of items which are to be subjected to a fixed (or nominal) level of stress and fairly constant environmental conditions during normal usage. If a required life $t_o$ is specified, then he will want to estimate the reliability at time $t_o$ after the stress is applied. If, on the other hand, a survival proportion $R$ is specified, then he will be interested in estimating the reliable life, or the time after the application of the stress at which $100(1-R)\%$ of the population of items will have failed.

He may know from theoretical considerations the family of distributions to which the failure times associated with the population of items belong. If so, he can conduct a life test by applying the nominal level of stress under appropriate environmental conditions to a sample chosen from the population of items and thus he can obtain an efficient estimate of the parameter of interest as a function of estimates of the distribution parameters (parameters of the failure-time distribution). These estimates will be based on the failure times observed during the life test.

Suppose, however, that the failure characteristics of the population are such that an excessively long period of time is required before even a single failure will occur at the nominal stress level. In such a case, the life test can be conducted at levels above nominal stress to obtain the failures required for estimation of the parameters, provided the form of the relationship between the parameters and the stress level is known. In the following discussion, the form of both this relationship and the failure-time distribution are assumed to be known from theory associated with the composition of the items.

It is assumed that in a life-testing situation the population failure times are identically distributed according to a given two-parameter Weibull law,

$$F(t) = \begin{cases} 1 - e^{-\left(\frac{t}{\xi}\right)^{\frac{1}{b}}} & , t \geq 0 \\ 0 & , \text{otherwise} \end{cases} \quad ; \ \xi, b > 0 , \qquad (1)$$

when the material and shape of the items tested, the environmental conditions of the test and the levels of stresses are fixed. We also assume that the size $n$ sample which is tested has been randomly selected from a population of items composed of homogeneous material and that all environmental conditions are held fixed save level of stress $\sigma$ for one type of stress of interest. Further, it is assumed, in agreement with [1], [8], and [9], that under these conditions the Weibull shape parameter $b$ is constant and that only the scale parameter $\xi$ changes as level of stress changes.

If the random variable $T$ represents failure time, then $X$, the natural logarithm of $T$, has the Type I extreme-value distribution of smallest values with location parameter $u$ (the natural logarithm of $\xi$) and scale parameter $b$, with $\pi^2 b^2/6$ the variance of $X$. We suppose that a given log failure time is a polynomial function of degree $k$ of $1/\sigma$ (a special case of this assumption, $\log t = \beta_0 + \beta_1/\sigma$, is given in [1]). Finally, it

is assumed that the $n_\alpha$ failure times observable at the $\alpha^{th}$ of $k$ experimental (or testing) stress levels are independent of those at any other of the $k - 1$ testing levels. This implies that the sample items are selected randomly for assignment to a stress level.

Under these assumptions it is desired to design the life test so that the least-squares curve (with expectation $x_{1/\sigma,b}$) has minimum variance at $\sigma = \frac{1}{\eta}$ when testing of the $n$ sample items is conducted in the stress domain $c \leq \sigma \leq d$ and $\frac{1}{\eta} < c$. We define this to be optimum estimation for this situation, which is one of testing at levels above nominal stress. Because this is an over-stress situation, it is assumed here that the stress levels at which testing will be conducted are sufficiently high so that all items can be allowed to fail, with no consideration necessary for waiting time for failure. This over-stress problem is equivalent to one of obtaining an optimum design for linear estimation of the polynomial function $x_{1/\sigma,b}$ at a point outside the experimental range of the independent variable.

Specifically, it is necessary to determine (a) the value for $k$, the number of sample points, (b) where in the domain the $k$ sample points should be assigned, and (c) the values of $n_\alpha$, $\alpha = 0, 1, \ldots, k - 1$, such that estimation is optimum for a given sample size $n$. The analysis involves an approximation in terms of $n_\alpha$, $\alpha = 0, 1, \ldots, k - 1$, of the

variance of the least-squares curve at $\sigma = \frac{1}{\eta}$. Hence the optimal solution given is approximate.

It will be clear from the model given below that the results of Hoel and Levine in [2], which are applicable to the case in which the observations are independent, have equal variances, and are selected from a Gaussian distribution, cannot be used directly to provide the design for optimum estimation here.

## THE MODEL

The derivations immediately following apply to the general case in which only the first $m_\alpha$ failure times, $m_\alpha \le n_\alpha$, may be observed, so the assumptions $m_\alpha = n_\alpha$ will not be employed until later. Let $X_{i,\alpha}$, $X_{1,\alpha} \le X_{2,\alpha} \le \dots \le X_{m_\alpha,\alpha} \le \dots \le X_{n_\alpha,\alpha}$, be the logarithm of the $i^{th}$ observed ordered failure time at the $\alpha^{th}$ stress level and let

$$Y_{i,\alpha} = \frac{X_{i,\alpha} - u_\alpha}{b}, \quad i = 1, 2, \dots, m_\alpha,$$

where $u_\alpha$ represents the value of $u\left(\frac{1}{\sigma}\right) = \beta_0 + \beta_1\left(\frac{1}{\sigma}\right)^2 + \dots + \beta_k\left(\frac{1}{\sigma}\right)^k$ at the $\alpha^{th}$ stress level, $\alpha = 0, 1, \dots, \ell - 1$. Then $Y_{i,\alpha}$ is the $i^{th}$ order statistic, $i = 1, 2, \dots, m_\alpha$, of a sample of size $n_\alpha$, $\alpha = 0, 1, \dots, \ell - 1$, from the distribution of $Y = \frac{X-u}{b}$, the

reduced first asymptotic distribution of smallest values.

Therefore,

$$E\left[X_{i,\alpha}\right] = u_\alpha + b\,E\left(Y_{i,\alpha}\right) \qquad (2)$$

$$i = 1, 2, \ldots, m_\alpha ,$$

$$\alpha = 0, 1, \ldots, \ell - 1 ,$$

and $\Sigma_X$, the covariance matrix of the $X$'s, is given by

$$\Sigma_X = b^2 V . \qquad (3)$$

Since it is assumed that the failure times at any given testing level are independent of those at any other level,

$$V = b^2 \begin{pmatrix} V_0 & & & \\ & V_1 & & \\ & & \ddots & \\ & & & V_{\ell-1} \end{pmatrix} , \qquad (4)$$

where $V_\alpha$, $\alpha = 0, 1, \ldots, \ell-1$, is the known covariance matrix of the first $m_\alpha$ of $n_\alpha$ order statistics of the distribution of $Y$. From (2), (3), and (4), it can be seen that the generalized Gauss-Markov Theorem, given in [4], applies. Hence, least-squares estimators having uniformly minimum variance among unbiased linear estimators can be obtained for $u_\alpha$ and $b$ when $\ell$, $m_\alpha$, $\alpha = 0, 1, \ldots, \ell-1$, and testing levels for stress are fixed. The problem of determining these quantities for optimum estimation is now considered.

## DESIGN FOR OPTIMUM ESTIMATION

The information matrix of the Gauss-Markov least-squares curve for any selection of $n$ observations in the interval $c \le \sigma \le d$ can be duplicated by choosing at most $k + 1$ distinct stress levels in this interval and allocating the $n$ observations appropriately to these new levels. We therefore let the number of stress levels (testing levels for the stress) be $k + 1$, and the $\alpha^{th}$ testing level be $\sigma_\alpha$, $\alpha = 0, 1, \ldots, k$, with $\sigma_0 < \sigma_1 < \ldots < \sigma_k$.

If $b$ is not known, it is necessary that $m_\alpha$ be at least 2 for at least one stress level in order that $u_\alpha$ be estimable (an unbiased estimator exist), $\alpha = 0, 1, \ldots, k$. This is true because $E\left(X_{1,\alpha}\right)$ is not equal to $u_\alpha$, but rather $E\left(X_{1,\alpha}\right) = u_\alpha + b\,E\left(Y_{1,\alpha}\right)$, with $E\left(Y_{1,\alpha}\right) \ne 0$, $\alpha = 0, 1, \ldots, k$. Note that if $Z_{i,n}$ is the $i^{th}$ order statistic, $i = 1, 2, \ldots, n$, of a size $n$ sample from a reduced Gaussian distribution, $E\left(Z_{1,1}\right) = 0$. If $b$ is known, $m_\alpha$ can be $1$, $\alpha = 0, 1, \ldots, k$. Optimum estimation for $b$ known will be considered in the section following this one.

Let $v^{i,j,\alpha}$ be the element in the $\alpha + i^{th}$ row and the $\alpha + j^{th}$ column, $i, j = 1, 2, \ldots, m_\alpha$, $\alpha = 0, 1, \ldots, k$, of $V^{-1}$, the inverse matrix of $V$. It can easily be shown that when $m_\alpha$ observations are made at each of $k + 1$ points, the least-squares estimator $X^*\left(\frac{1}{\sigma}\right)$ of the expected value of the least-squares curve at $\left(\frac{1}{\sigma}\right)$,

$x_{1/\sigma,b} = \beta_0 + \beta_1\left(\frac{1}{\sigma}\right) + \cdots + \beta_k\left(\frac{1}{\sigma}\right)^k + \kappa_0(b)$ (with $\kappa_0(b)$ defined below), passes through

$$\frac{1}{\sigma_\alpha}, \frac{\sum_{j=1}^{m_\alpha} \sum_{i=1}^{m_\alpha} v^{i,j,\alpha} X_{i,\alpha}}{\sum_{j=1}^{m_\alpha} \sum_{i=1}^{m_\alpha} v^{i,j,\alpha}} \tag{5}$$

and can be expressed in the form

$$X^*\left(\frac{1}{\sigma}\right) = \sum_{\alpha=0}^{k} L_\alpha\left(\frac{1}{\sigma}\right) \frac{\sum_{j=1}^{m_\alpha} \sum_{i=1}^{m_\alpha} v^{i,j,\alpha} X_{i,\alpha}}{\sum_{j=1}^{m_\alpha} \sum_{i=1}^{m_\alpha} v^{i,j,\alpha}} = \sum_{\alpha=0}^{k} L_\alpha\left(\frac{1}{\sigma}\right) X^*\left(\frac{1}{\sigma_\alpha}\right), \tag{6}$$

where $L_\alpha\left(\frac{1}{\sigma}\right)$ is the Lagrange polynomial

$$\frac{\left(\frac{1}{\sigma} - \frac{1}{\sigma_0}\right) \cdots \left(\frac{1}{\sigma} - \frac{1}{\sigma_{\alpha-1}}\right)\left(\frac{1}{\sigma} - \frac{1}{\sigma_{\alpha+1}}\right) \cdots \left(\frac{1}{\sigma} - \frac{1}{\sigma_k}\right)}{\left(\frac{1}{\sigma_\alpha} - \frac{1}{\sigma_0}\right) \cdots \left(\frac{1}{\sigma_\alpha} - \frac{1}{\sigma_{\alpha-1}}\right)\left(\frac{1}{\sigma_\alpha} - \frac{1}{\sigma_{\alpha+1}}\right) \cdots \left(\frac{1}{\sigma_\alpha} - \frac{1}{\sigma_k}\right)}, \tag{7}$$

$$\alpha = 0, 1, \ldots, k.$$

This represents a generalization of the better known result used by Hoel and Levine in [2], where $m_\alpha = n_\alpha$ and the model is

$$E\left[W_{i,\alpha}\right] = u_\alpha = \beta_0 + \beta_1(z_\alpha) + \cdots + \beta_k(z_\alpha)^k, \tag{8}$$

$$i = 1, 2, \ldots, n_\alpha,$$

$$\alpha = 0, 1, \ldots, \ell-1,$$

with

$$\Sigma_W = b^2 I , \tag{9}$$

and I the identity matrix. Here, $W_{i,\alpha}$ is the $i^{th}$ Gaussian order statistic, $i = 1, 2, \ldots, n_\alpha$, $\alpha = 0, 1, \ldots, \ell - 1$, and for $\ell = k + 1$, the least-squares estimator $W^*$ of $w = \sum_{j=0}^{k} \beta_j z^j$ passes through $z_\alpha$, $\bar{w}_\alpha$. $W^*$ can be expressed in the form

$$\sum_{\alpha=0}^{k} L_\alpha(z)\bar{w}_\alpha ,$$

with $\bar{w}_\alpha$ equal to $\frac{1}{n_\alpha} \sum_{i=1}^{n_\alpha} W_{i,\alpha}$, $\alpha = 0, 1, \ldots, k$, and $L_\alpha(z)$ defined by (7). It might be noted that $\sum_{\alpha=0}^{k} L_\alpha(z) = 1$ .

The variance $\mathrm{Var}\left(X^*\left(\frac{1}{\sigma}\right)\right)$ of the least-squares estimator $X^*\left(\frac{1}{\sigma}\right)$ of $x_{1/\sigma,b}$ is simply the sum of the variances at the $k + 1$ testing levels, or

$$\mathrm{Var}\left(X^*\left(\frac{1}{\sigma}\right)\right) = b^2 \sum_{\alpha=0}^{k} L_\alpha^2\left(\frac{1}{\sigma}\right) \frac{\sum_{j=1}^{m_\alpha} \sum_{i=1}^{m_\alpha} v_{i,j,\alpha} \sum_{r=1}^{m_\alpha} v^{i,r,\alpha} \sum_{s=1}^{m_\alpha} v^{s,j,\alpha}}{\left(\sum_{j=1}^{m_\alpha} \sum_{i=1}^{m_\alpha} v^{i,j,\alpha}\right)^2}$$

which reduces to

$$\mathrm{Var}\left(X^*\left(\frac{1}{\sigma}\right)\right) = b^2 \sum_{\alpha=0}^{k} L_\alpha^{\,2}\left(\frac{1}{\sigma}\right) \Big/ \sum_{j=1}^{m_\alpha} \sum_{i=1}^{m_\alpha} v^{i,j,\alpha} . \tag{10}$$

Here, $v_{i,j,\alpha}$ is the element in the $\alpha + i^{th}$ row and the $\alpha + j^{th}$ column, $i, j = 1, 2, \ldots, m_\alpha$, $\alpha = 0, 1, \ldots, k$, of $v$.

Let $u^*\left(\frac{1}{\sigma_\alpha}\right)$ be

$$\frac{-\sum_{s=1}^{m_\alpha}\sum_{r=1}^{m_\alpha}\left[\sum_{j=1}^{m_\alpha}\sum_{i=1}^{m_\alpha}E\left(Y_{i,\alpha}\right)v^{i,j,\alpha}E\left(Y_{r,\alpha}\right)v^{i,s,\alpha}-\sum_{j=1}^{m_\alpha}\sum_{i=1}^{m_\alpha}E\left(Y_{i,\alpha}\right)E\left(Y_{j,\alpha}\right)v^{i,j,\alpha}v^{r,s,\alpha}\right]X_{s,\alpha}}{\Delta}$$

and $b_\alpha^*$ be

$$\frac{\sum_{s=1}^{m_\alpha}\sum_{r=1}^{m_\alpha}\left[\sum_{j=1}^{m_\alpha}\sum_{i=1}^{m_\alpha}v^{i,j,\alpha}E\left(Y_{r,\alpha}\right)v^{r,s,\alpha}-\sum_{j=1}^{m_\alpha}\sum_{i=1}^{m_\alpha}E\left(Y_{i,\alpha}\right)v^{i,j,\alpha}v^{r,s,\alpha}\right]X_{s,\alpha}}{\Delta},$$

where

$$\Delta = \sum_{j=1}^{m_\alpha}\sum_{i=1}^{m_\alpha}v^{i,j,\alpha}\sum_{j=1}^{m_\alpha}\sum_{i=1}^{m_\alpha}E\left(Y_{i,\alpha}\right)E\left(Y_{j,\alpha}\right)v^{i,j,\alpha} - \left(\sum_{j=1}^{m_\alpha}\sum_{i=1}^{m_\alpha}E\left(Y_{i,\alpha}\right)v^{i,j,\alpha}\right)^2 .$$

The expressions $u^*\left(\frac{1}{\sigma_\alpha}\right)$ and $b_\alpha^*$ are the least-squares uniformly minimum-variance unbiased linear estimators of $u\left(\frac{1}{\sigma}\right)$ and b at the $\alpha^{th}$ sample point. Let the variance of the $b_\alpha^*$ be $\mathrm{Var}\left(b_\alpha^*\right)$, the variance of $u^*\left(\frac{1}{\sigma_\alpha}\right)$ be $\mathrm{Var}\left(u^*\left(\frac{1}{\sigma_\alpha}\right)\right)$, and the covariance of $u^*\left(\frac{1}{\sigma_\alpha}\right)$ and $b_\alpha^*$ be $\mathrm{Cov}\left(u^*\left(\frac{1}{\sigma_\alpha}\right), b_\alpha^*\right)$. Then $\mathrm{Var}\left(X^*\left(\frac{1}{\sigma_\alpha}\right)\right)$, as given by (10), can be shown to be equal to

$$\left[\mathrm{Var}\left(u^*\left(\frac{1}{\sigma_\alpha}\right)\right) - \mathrm{Cov}^2\left(u^*\left(\frac{1}{\sigma_\alpha}\right), b_\alpha^*\right) / \mathrm{Var}\left(b_\alpha^*\right)\right], \tag{11}$$

and it can also be shown that

$$X^*\left(\frac{1}{\sigma_\alpha}\right) = \left[u^*\left(\frac{1}{\sigma_\alpha}\right) - \mathrm{Cov}\left(u^*\left(\frac{1}{\sigma_\alpha}\right), b_\alpha^*\right) b_\alpha^* / \mathrm{Var}\left(b_\alpha^*\right)\right], \tag{12}$$

for $m_\alpha \geq 2$, $\alpha = 0, 1, \ldots, k$. For $m_\alpha = 1$, $X^*\left(\frac{1}{\sigma_\alpha}\right)$ is simply $X_{1,\alpha}$, $\alpha = 0, 1, \ldots, k$, with variance $\pi^2 \frac{b^2}{6}$. It can be seen from (12) that $\varkappa_0(b)$ in $x_{1/\sigma, b} = u\left(\frac{1}{\sigma}\right) + \varkappa_0(b)$ is equal[1] to $-\sum_{\alpha=0}^{k} L_\alpha\left(\frac{1}{\sigma}\right) \frac{\operatorname{Cov}\left(u^*\left(\frac{1}{\sigma_\alpha}\right), b^*_\alpha\right)}{\operatorname{Var}\left(b^*_\alpha\right)} b$.

The expression (11) also defines, for $m_\alpha \geq 2$, the variance of $\bar{w}_\alpha = \sum_{i=0}^{n_\alpha} W_{i,\alpha}$, described by (8) and (9) with $z_\alpha = \frac{1}{\sigma_\alpha}$ and $m_\alpha = n_\alpha$, where $u^*\left(\frac{1}{\sigma_\alpha}\right)$ and $b^*_\alpha$ are the least-squares best linear unbiased estimators of $u$ and $b$, respectively, at the $\alpha^{th}$ sample point, $\alpha = 0, 1, \ldots, k$. When the density function of $\bar{w}_\alpha$ is symmetric about the location parameter $u$, however, as when $W_{i,\alpha}$, $i = 0, 1, \ldots, n_\alpha$, $\alpha = 0, 1, \ldots, k$ is a Gaussian order statistic, then the covariance between the least-squares estimators $u^*\left(\frac{1}{\sigma_\alpha}\right)$ and $b^*_\alpha$ of the location parameter $u_\alpha$, $\alpha = 0, 1, \ldots, k$, and the scale parameter $b$, respectively, is zero, as is shown in [4]. In this case the variance of $\bar{w}_\alpha$ is simply equal to the variance of $u^*\left(\frac{1}{\sigma_\alpha}\right)$, as in [2], where $u^*\left(\frac{1}{\sigma_\alpha}\right) = \bar{w}_\alpha$, $\alpha = 0, 1, \ldots, k$.

For $m_\alpha = 1$, $\operatorname{Var}\left(X^*\left(\frac{1}{\sigma_\alpha}\right)\right) = \operatorname{Var}\left(X_{1,\alpha}\right) = \pi^2 b^2/6 \doteq 1.64493\ b^2$. Also it is possible for any given combination of values of $m_\alpha$ and $n_\alpha$, $2 \leq m_\alpha \leq n_\alpha$, to calculate a value for the variance of $X^*\left(\frac{1}{\sigma_\alpha}\right)$ using (11) and the tables

[1] Where any $m_\alpha$ is equal to 1, $-\frac{\operatorname{Cov}\left(u^*\left(\frac{1}{\sigma_\alpha}\right), b^*_\alpha\right)}{\operatorname{Var}(b^*_\alpha)}$ is taken to be $E(Y_{1,\alpha})$, $\alpha = 0, 1, \ldots, k$.

in [3] for $n_\alpha \le 6$, [10] for $n_\alpha \le 20$, or [6] for $n_\alpha \le 25$. For $m_\alpha = n_\alpha$, one obtains (to within a unit in the fifth decimal place),

| $n_\alpha$ | $\text{Var } X^*\left(\frac{1}{\sigma_\alpha}\right)$ |
|---|---|
| 1 | $1.64493 \ b^2$ |
| 2 | $.65373 \ b^2$ |
| 3 | $.40108 \ b^2$ |
| 4 | $.28810 \ b^2$ |
| . | . |
| . | . |
| . | . |
| $n_\alpha \to \infty$ | $b^2/n_\alpha$, $\alpha = 0, 1, \ldots, k.$ |

A general expression for $\text{Var } X^*\left(\frac{1}{\sigma_\alpha}\right)$ in terms of $n_\alpha$, for $m_\alpha = n_\alpha$, can then be given approximately (to almost within a unit in the second significant figure) by $b^2/(n_\alpha - 0.43)$, $\alpha = 0, 1, \ldots, k$.

Let $m_\alpha = n_\alpha$ and $p_\alpha = n_\alpha/n$, $\alpha = 0, 1, \ldots, k$, so that $\sum_{\alpha=0}^{k} p_\alpha = 1$.

The variance of the least-squares estimator $X^*(\eta)$ at the stress level $1/\eta$ is of the approximate form

$$(b^2/n) \sum_{\alpha=0}^{k} L_\alpha^2(\eta)/(p_\alpha - 0.43/n). \tag{13}$$

For any choice of testing levels $\sigma_0, \sigma_1, \ldots, \sigma_k$, the approximate variance given by (13) will be minimized, under the constraint $\sum_{\alpha=0}^{k} p_\alpha = 1$, with $0 \le p_\alpha \le 1$ and with $\frac{1}{\eta} \ne \sigma_\alpha$, if

$$p_\alpha = \frac{(1 - 0.43(k+1)/n)\,|L_\alpha(\eta)|}{\sum_{j=0}^{k} |L_j(\eta)|} + \frac{0.43}{n}, \quad \alpha = 0, 1, \ldots, k. \tag{14}$$

This result is obtained, as is Lemma 1 of [2], simply by standard calculus techniques. If the proportion $p_\alpha$ of the sample assigned to the stress level $\sigma_\alpha$ is given by (14), the variance of the least-squares estimator is approximately

$$(b^2/n) \left( \sum_{\alpha=0}^{k} |L_\alpha(\eta)| \right)^2 / (1 - 0.43(k+1)/n) \tag{15}$$

for $\eta$ as defined.

Clearly, the expression (15) will be minimized if $\sum_{\alpha=0}^{k} |L_\alpha(\eta)|$ is minimized. Therefore, the theorem below follows immediately from the proof by Hoel and Levine (consisting of a minimization of $\sum_{\alpha=0}^{k} |L_\alpha(\eta)|$, $\eta > 1/c$) of a similar theorem in [2] for the model given by (8) and (9) plus the normality assumption.

THEOREM I. Under the assumptions (2), (3), and (4), let $p_\alpha$, given by formula (14), be the proportion of the sample of $n$ items tested at the stress level $\sigma_\alpha$, $\sigma_o \leq \sigma_1 \leq \ldots \leq \sigma_k$, in the range $c \leq \sigma_\alpha \leq d$, $\alpha = 0, 1, \ldots, k$, and let $1/\eta$ be less than $c$. The value of $\frac{1}{\sigma_\alpha}$ that will minimize the expression (15) giving the approximate variance of $X^*(\eta)$, the point on the least-squares curve at $\sigma = 1/\eta$, is

$$\frac{1}{\sigma_\alpha} = \left\{- \cos\left[(k - \alpha)\pi/k\right]\right\} \left(\frac{1}{c} - \frac{1}{d}\right)/2 + \left(\frac{1}{c} + \frac{1}{d}\right)/2 \text{ , } \alpha = 0, 1, \ldots, k. \tag{16}$$

Theorem I then gives the solution to the problem of obtaining the least-squares curve with minimum-variance intercept at $\sigma = 1/\eta$ under the assumptions (2), (3), and (4). In the following section, properties of this minimum-variance intercept and the least-squares curve in general will be considered.

## PROPERTIES OF THE LEAST-SQUARES CURVE UNDER THE GENERAL MODEL

An interesting property of the least-squares curve which holds at the $k + 1$ testing levels is first demonstrated. Consider any distribution with unknown location-scale parameter $\theta = (\theta_1, \theta_2)$ such that $f_\theta(x) = \frac{1}{\theta_2} g\left(\frac{x-\theta_1}{\theta_2}\right)$ for some $g$. Let $\theta_1^*$ and $\theta_2^*$ be the unique (with probability 1) uniformly minimum-variance unbiased estimators of $\theta_1$ and $\theta_2$, respectively, in some class of estimators (linear estimators, for example, or all possible

estimators). Let $A\theta_2^2$ and $C\theta_2^2$ be the variances of $\theta_1^*$ and $\theta_2^*$, respectively, and $B\theta_2^2$ their covariance, and let loss be squared error divided by $\theta_2^2$. It is shown in [7] that (1) for any given function of the form $\Phi = \ell_1 \theta_1 + \ell_2 \theta_2$ the unique estimator with smallest expected loss among estimators in the class invariant under transformations of location and scale (when the best unbiased estimator of $\Phi$ in the class is $\ell_1 \theta_1^* + \ell_2 \theta_2^*$) is

$$\tilde{\Phi} = \ell_1 \theta_1^* + (\ell_2 - \ell_1 B)\, \theta_2^*/(1+C) \tag{17}$$

and (2) if $\tilde{\Phi}$ is linear in the observed x's, then it is the unique admissible[1] minimax linear estimator of $\Phi$. The proof of (1) is very similar to the derivation in [6] of the expression for Cramér-Rao type bounds for invariant estimators of location parameters of the form $\theta_1 + \ell\theta_2$.

Let $\mathrm{Var}\left(u^*\left(\frac{1}{\sigma_\alpha}\right)\right)$ be $A_\alpha b^2$, $\mathrm{Var}(b_\alpha^*)$ be $C_\alpha b^2$, and $\mathrm{Cov}\left(u^*\left(\frac{1}{\sigma_\alpha}\right), b_\alpha^*\right)$ be $B_\alpha^2 b^2$. Then, by (17), the unique admissible[1] minimax linear estimators of $u_\alpha$ and $b$ at the $\alpha^{th}$ stress level are $\tilde{u}\left(\frac{1}{\sigma_\alpha}\right) = u^*\left(\frac{1}{\sigma_\alpha}\right) - B_\alpha b_\alpha^*/(1+C_\alpha)$ and $\tilde{b}_\alpha = b_\alpha^*/(1+C_\alpha)$, respectively, $\alpha = 0, 1, \ldots, k$. Furthermore, it can be very readily seen that $X^*\left(\frac{1}{\sigma_\alpha}\right) = u^*\left(\frac{1}{\sigma_\alpha}\right) - B_\alpha b_\alpha^*/C_\alpha$ can also be expressed as $X^*\left(\frac{1}{\sigma_\alpha}\right) = \tilde{u}\left(\frac{1}{\sigma_\alpha}\right) - B_\alpha \tilde{b}_\alpha/C_\alpha$, and that $X^*\left(\frac{1}{\sigma_\alpha}\right)$ is the best

[1]The proof of linear admissibility for $\tilde{\Phi}$, which had been shown in [5] to be the best linear invariant estimator of $\Phi$, was outlined in detail to this author by M. R. Mickey of the University of California at Los Angeles by personal communication.

linear invariant estimator of $x_{1/\sigma_\alpha, b}$, its expected value, $\alpha = 0, 1, \ldots, k$. This result is related to the fact that $X^*\left(\frac{1}{\sigma_\alpha}\right)$ and $b_\alpha^*$ have zero covariance, $\alpha = 0, 1, \ldots, k$.

Another property of the least-squares curve which will be demonstrated below is the following. Under the assumptions (2), (3), and (4), there exists no linear combination of the ordered observations at stress level $\sigma_\alpha$ and no point $u_\alpha + \zeta b$, $-\infty < \zeta < \infty$, along the $\frac{1}{\sigma_\alpha}$ ordinate such that the expected squared deviation of this linear combination from $u_\alpha + \zeta b$ is smaller than that of $X^*\left(\frac{1}{\sigma_\alpha}\right)$, the intercept of the least-squares curve at $\sigma = \sigma_\alpha$, from its expected value, $x_{1/\sigma_\alpha, b} = \beta_0 + \beta\left(\frac{1}{\sigma_\alpha}\right) + \ldots + \beta_k\left(\frac{1}{\sigma_\alpha}\right)^k + x_0(b)$, for at least one combination of values of $u_\alpha$ and $b$ and at least as small for all combinations of values of $u_\alpha$ and $b$, $\alpha = 0, 1, \ldots, k$. We say then that the "admissibility" of the linear combination $X^*\left(\frac{1}{\sigma_\alpha}\right)$ will be demonstrated. It will also be shown that when loss is squared error divided by $b^2$, $X^*\left(\frac{1}{\sigma_\alpha}\right)$ is a minimax linear combination of the observations (has smaller maximum expected loss from $x_{1/\sigma_\alpha, b}$ as a function of $u_\alpha$ and $b$ than any other linear combination from any point $u_\alpha + \zeta b, -\infty < \zeta < \infty$, along the $\frac{1}{\sigma_\alpha}$ ordinate), $\alpha = 0, 1, \ldots, k$. Properties of the least-squares curve at $\sigma = \frac{1}{\eta}$ will also be demonstrated.

Let $X_T^*(\eta)$ be the intercept on the least-squares curve at $\sigma = \frac{1}{\eta}$ when $n$ observations are made at $\sigma_1 \le \sigma_2 \le \ldots \le \sigma_k$, with $c \le \sigma_\alpha \le d$ and $\frac{1}{\eta} < c$, and when $\sigma_\alpha$ and $p_\alpha$, the proportion of the $n$ failure times observed at the $\alpha^{th}$ level, are those specified by Theorem I. Assume (13) gives the exact variance of $X^*(\eta)$, defined by (10). The following theorem will now be proved.

THEOREM II. Let loss be squared error divided by $b^2$. Under the assumptions (2), (3) and (4), $X^*\left(\frac{1}{\sigma_\alpha}\right)$, the intercept of the least-squares curve at $\frac{1}{\sigma_\alpha}$, is the unique admissible minimax linear combination of the ordered observations made at $\sigma = \sigma_\alpha$. $X_T^*(\eta)$, defined above, is the unique admissible minimax linear combination of $n$ observations extrapolated to $\sigma = \frac{1}{\eta}$, when it is assumed that (15) gives its exact variance. $X_T^*(\eta)$ is also the linear function with uniformly smallest expected loss of best linear invariant estimators of $u$ and $b$ at $k + 1$ stress levels at which life testing may occur, extrapolated to $\sigma = \frac{1}{\eta}$ (when it is assumed that (15) gives the exact variance of $X_T^*(\eta)$).

PROOF

Each of the statements in the demonstration of the proof of the first statement of Theorem II holds for $\alpha = 0, 1, \ldots, k$. Let us suppose that there exists a linear function $\tilde{X}\left(\frac{1}{\sigma_\alpha}\right)$ of the ordered observations

$X_{1,\alpha}, X_{2,\alpha} \ldots, X_{n_\alpha,\alpha}$ and a point $u_\alpha + \zeta b$, $-\infty < \zeta < \infty$, along the $\frac{1}{\sigma_\alpha}$ ordinate, such that the expected squared deviation of $\bar{X}\left(\frac{1}{\sigma_\alpha}\right)$ from $u_\alpha + \zeta b$ is at least as small as that of $X^*\left(\frac{1}{\sigma_\alpha}\right)$ from $x_{1/\sigma_\alpha, b}$ for all combinations of values of $u_\alpha$ and $b$ and smaller for at least one combination of values of $u_\alpha$ and $b$. Hence it is supposed that $X^*\left(\frac{1}{\sigma_\alpha}\right)$ is an inadmissible linear combination of the ordered observations.

Suppose that $\bar{X}\left(\frac{1}{\sigma_\alpha}\right)$ is not of the form $\ell_1 u^*\left(\frac{1}{\sigma_\alpha}\right) + \ell_2 b^*_\alpha$. Then by the generalized Gauss-Markov Theorem [4] there exists $\bar{\bar{X}}\left(\frac{1}{\sigma_\alpha}\right)$ of the form $\ell_1 u^*\left(\frac{1}{\sigma_\alpha}\right) + \ell_2 b^*_\alpha$ such that $E\left(\bar{\bar{X}}\left(\frac{1}{\sigma_\alpha}\right)\right) = E\left(\bar{X}\left(\frac{1}{\sigma_\alpha}\right)\right)$ and $\mathrm{Var}\left(\bar{\bar{X}}\left(\frac{1}{\sigma_\alpha}\right)\right) < \mathrm{Var}\left(\bar{X}\left(\frac{1}{\sigma_\alpha}\right)\right)$ for all $u_\alpha$ and $b$. Thus,

$$\mathrm{Var}\left(\bar{\bar{X}}\left(\frac{1}{\sigma_\alpha}\right)\right) + \left[E\left(\bar{\bar{X}}\left(\frac{1}{\sigma_\alpha}\right)\right) - (u_\alpha - \zeta b)\right]^2 < \mathrm{Var}\left(\bar{X}\left(\frac{1}{\sigma_\alpha}\right)\right) + \left[E\left(\bar{X}\left(\frac{1}{\sigma_\alpha}\right)\right) - (u_\alpha - \zeta b)\right]^2$$

for all $u_\alpha$ and $b$. Hence, it may as well be supposed that

$$\bar{X}\left(\frac{1}{\sigma_\alpha}\right) = S_{1,\alpha}\, u^*\left(\frac{1}{\sigma_\alpha}\right) + S_{2,\alpha}\, b^*_\alpha .$$

Then the expected loss function for $\bar{X}\left(\frac{1}{\sigma_\alpha}\right)$ is

$$S^2_{1,\alpha} A_\alpha + 2S_{1,\alpha} S_{2,\alpha} B_\alpha + S^2_{2,\alpha} C_\alpha + \left[(S_{1,\alpha} - 1)u_\alpha + (S_{2,\alpha} - \zeta)b\right]^2 / b^2$$

which is assumed to be less than $A_\alpha - B^2_\alpha / C_\alpha$ for all $u_\alpha$ and $b$. The value of $S_{1,\alpha}$ must then be equal to 1 if this inequality is to hold identically in $u_\alpha$ and $b$.

Thus, it is assumed that $\bar{X}\left(\frac{1}{\sigma_\alpha}\right) = u^*\left(\frac{1}{\sigma_\alpha}\right) + S_{2,\alpha} b_\alpha^*$ with expected loss given by

$$A_\alpha + 2S_{2,\alpha} B_\alpha + S_{2,\alpha}^2 C_\alpha + \left(S_{2,\alpha} - \zeta\right)^2 \tag{18}$$

The expression (18) is minimized with respect to $\zeta$ and $S_{2,\alpha}$ jointly when $\zeta = S_{2,\alpha} = -B_\alpha / C_\alpha$. Hence $X^*\left(\frac{1}{\sigma_\alpha}\right)$ is an admissible linear combination of the observations at the $\alpha^{th}$ stress level, $\alpha = 0, 1, \ldots, k$. The proof of admissibility for $X^*\left(\frac{1}{\sigma_\alpha}\right)$, $\alpha = 0, 1, \ldots, k$, is similar to that referred to in the footnote on page 33. An alternative proof would begin with a demonstration that for every biased linear combination of the observations, there exists an unbiased combination with smaller risk (expected loss).

Suppose $X^*\left(\frac{1}{\sigma_\alpha}\right)$ is not a minimax-risk linear combination of the observations. Then there exists a linear combination having maximum risk smaller than that of $X^*\left(\frac{1}{\sigma_\alpha}\right)$, which has constant risk with respect to $u_\alpha$ and $b$, $\alpha = 0, 1, \ldots, k$, since loss is squared error divided by $b^2$. The existence of such an estimator would thus imply that $X^*\left(\frac{1}{\sigma_\alpha}\right)$, $\alpha = 0, 1, \ldots, k$, was inadmissible. Hence $X^*\left(\frac{1}{\sigma_\alpha}\right)$ is the unique (with probability 1) admissible minimax linear combination of the observations at the $\alpha^{th}$ stress level, $\alpha = 0, 1, \ldots, k$. The uniqueness follows from the uniqueness of $u^*\left(\frac{1}{\sigma_\alpha}\right)$ and $b_\alpha^*$, $\alpha = 0, 1, \ldots, k$.

Now, consider

$$X_T^*(\eta) = \sum_{\alpha=0}^{k} L_\alpha(\eta) \left[u^*\left(\frac{1}{\sigma_\alpha}\right) - B_\alpha b_\alpha^* / C_\alpha\right],$$

the least-squares intercept at $\sigma = \frac{1}{\eta}$, where $p_\alpha$, which represents the proportion of the n sample values tested at $\frac{1}{\sigma_\alpha}$, and $\frac{1}{\sigma_\alpha}$, $\alpha = 0, 1, \ldots, k$, are chosen according to the specifications of Theorem I. It is assumed that $c \le \sigma_\alpha \le d$, $\alpha = 0, 1, \ldots, k$, and $\frac{1}{\eta} < c$.

Let $\bar{X}(\eta) = \sum_{\alpha=0}^{k} L_\alpha(\eta) \sum_{i=1}^{n_\alpha} a_{i,\alpha} X_{i,\alpha}$, with $\sum_{\alpha=0}^{k} n_\alpha = n$, estimate a point $u(\eta) + \zeta b$, $-\infty < \zeta < \infty$, on the ordinate $\eta$. Suppose that $a_i$, $i = 1, 2, \ldots, n_\alpha$, $p_\alpha = n_{\alpha/n}$, and $\sigma_\alpha$, $\alpha = 0, 1, \ldots, k$, are chosen so that the mean squared deviation of $\bar{X}(\eta)$ from $u(\eta) + \zeta b$ is less than or equal to the variance of $X^*_\eta(\eta)$ for all b, $u(\eta)$, and $u_\alpha$, $\alpha = 0, 1, \ldots, k$. Suppose that the expectation of $\bar{X}(\eta)$ is not $u(\eta) + \zeta b$. Then the expected squared deviation of $\bar{X}(\eta)$ from its expected value is less than that of $\bar{X}(\eta)$ from $u(\eta) + \zeta b$. Hence, it is assumed that $E(\bar{X}(\eta)) = u(\eta) + \zeta b$ for all b, $u(\eta)$, and $u\left(\frac{1}{\sigma_\alpha}\right)$, $\alpha = 0, 1, \ldots, k$. Then when $\bar{X}(\eta)$ is of the form

$$\sum_{\alpha=0}^{k} L_\alpha(\eta) \left[u^*\left(\frac{1}{\sigma_\alpha}\right) + \tau_\alpha b^*_\alpha\right], \tag{19}$$

with variance

$$b^2 \sum_{\alpha=0}^{k} \left[L^2_\alpha(\eta)\left(A_\alpha + 2\tau_\alpha B_\alpha + \tau^2_\alpha C_\alpha\right)\right], \tag{20}$$

it has uniformly smaller variance than any other linear combination with uniform expectation $u(\eta) + \sum_{\alpha=0}^{k} L_\alpha(\eta)\tau_\alpha b = u(\eta) + \zeta b$ for any specified combination of testing levels and allocation of the sample to the $k + 1$ levels. If any $n_\alpha$ is equal to 1, $u^*\left(\frac{1}{\sigma_\alpha}\right) + \tau_\alpha b_\alpha^*$ is taken to be $X_{1,\alpha}$ with variance $\pi^2 b^2/6$, $\alpha = 0, 1, \ldots, k$.

The variance of $\tilde{X}(\eta)$ is minimized independently of the value of $p_\alpha$ and $\frac{1}{\sigma_\alpha}$, $\alpha = 0, 1, \ldots, k$ if $\tau_\alpha = - B_\alpha/C_\alpha$. Hence, $X^*(\eta)$ is admissible among linear combinations of $n$ sample observations when $\frac{1}{\sigma_\alpha}$ and $p_\alpha$, $\alpha = 0, 1, \ldots, k$, are fixed. Then by Theorem I and the reasoning applied in proving the first statement of this theorem, $X_T^*(\eta)$ is the unique admissible minimax linear combination of $n$ sample failure-time logarithms observed at $\sigma_1, \sigma_2, \ldots, \sigma_k$, $c \le \sigma_\alpha \le d$, $\alpha = 0, 1, \ldots, k$, and extrapolated to $\sigma = \frac{1}{\eta}, \frac{1}{\eta} < c$ (under the assumption that (15) gives the exact variance of $X^*(\eta)$).

On the basis of the preceding derivation and the assumption (concerning the variance of $X^*(\eta)$) it follows immediately that

$$X_T^*(\eta) = \sum_{\alpha=0}^{k} L_\alpha(\eta)\left[u^*\left(\frac{1}{\sigma_\alpha}\right) - \frac{B_\alpha b_\alpha^*}{(1+C_\alpha)} - \frac{B_\alpha b_\alpha^*}{C_\alpha(1+C_\alpha)}\right],$$ with expectation

$$\sum_{\alpha=0}^{k} L_\alpha(\eta)\left[u_\alpha - \frac{B_\alpha b}{1+C_\alpha} - \frac{B_\alpha b}{C_\alpha(1+C_\alpha)}\right] \text{ and variance } \sum_{\alpha=0}^{k} L_\alpha(\eta)\left[A_\alpha - B_\alpha^2/C_\alpha\right], \text{ is}$$

the linear combination with uniformly smallest expected loss of best linear invariant estimators of $u$ and $b$ at $k+1$ stress levels at which life testing may occur, extrapolated to $\sigma = \frac{1}{\eta}$. It is interesting to note that when the model is given by (8) and (9) plus the assumption of a Gaussian distribution for the observations,

$b^2 \sum_{\alpha=0}^{k} L_\alpha^2(\eta)/n_\alpha$ is the exact variance of $X^*(\eta) = \sum_{\alpha=0}^{k} L_\alpha \sum_{i=1}^{n_\alpha} x_{i,\alpha}\, n_\alpha$.

## A COMPARISON OF THE TRADITIONAL AND THE DERIVED DESIGNS

As an illustration of the result given by Theorem I, consider the following example. It is assumed that $u$ is of the form $\beta_0 + \beta_1\left(\frac{1}{\sigma}\right) + \beta_2\left(\frac{1}{\sigma}\right)^2 + \beta_3\left(\frac{1}{\sigma}\right)^3 + \beta_4\left(\frac{1}{\sigma}\right)^4$ and that 10 items are to be tested until failure over the stress interval $\frac{1}{3\sqrt{2}} \le \sigma \le \frac{1}{\sqrt{2}}$. Using a traditional design specifying equal numbers of items tested at equal intervals over the testing domain, one obtains the following.

| | | | | |
|---|---|---|---|---|
| $\frac{1}{\sigma_4} = \sqrt{2}$ | $\frac{1}{\sigma_3} = 3\sqrt{2}/2$ | $\frac{1}{\sigma_2} = 2\sqrt{2}$ | $\frac{1}{\sigma_1} = 5\sqrt{2}/2$ | $\frac{1}{\sigma_0} = 3\sqrt{2}$ |
| $n_4 = 2$ | $n_3 = 2$ | $n_2 = 2$ | $n_1 = 2$ | $n_0 = 2$ |

For optimum estimation at $\sigma = \frac{1}{4\sqrt{2}}$, the theorem specifies approximately the design given below.

$$\frac{1}{\sigma_4} = \sqrt{2} \qquad \frac{1}{\sigma_3} = 2\sqrt{2}-1 \qquad \frac{1}{\sigma_2} = 2\sqrt{2} \qquad \frac{1}{\sigma_1} = 2\sqrt{2}+1 \qquad \frac{1}{\sigma_0} = 3\sqrt{2}$$

$$n_4 = 1 \qquad n_3 = 2 \qquad n_2 = 2 \qquad n_1 = 3 \qquad n_0 = 2 .$$

The least-squares curves for the two sets of 10 observations have variances at $\sigma = \frac{1}{4\sqrt{2}}$ equal to approximately (to the nearest integer) 2912 $b^2$ for the traditional design and 1238 $b^2$ for the derived design specified by the theorem.

It is shown in [7] that for $b$ known, the best linear invariant estimator of $u_\alpha$, for $n_\alpha \geq 2$, is $\hat{u}\left(\frac{1}{\sigma_\alpha}\right) = u^*\left(\frac{1}{\sigma_\alpha}\right) - \left(B_\alpha/C_\alpha\right)\left(b_\alpha^* - b\right)$, with mean squared error (variance, since $\hat{u}\left(\frac{1}{\sigma_\alpha}\right)$ is unbiased) $\left[A_\alpha - B_\alpha^2/C_\alpha\right] b^2$, $\alpha = 0, 1, \ldots, k$. The variance values given above for the two designs are, therefore, also appropriate for $\hat{u}(\eta) = \sum_{\alpha=0}^{k} L_\alpha(\eta)\left[X^*\left(\frac{1}{\sigma_\alpha}\right) + \left(B_\alpha/C_\alpha\right)b\right]$, with $B_\alpha/C_\alpha$ taken to be $-E\left(Y_{1,\alpha}\right)$ for any $n_\alpha$ equal to 1, $\alpha = 0, 1, \ldots, k$.

If $b$ is not known, then the best linear unbiased estimator of this parameter based on any given design can easily be shown to be $\sum_{\alpha=0}^{k} \frac{b_\alpha^*/C_\alpha}{\sum_{\alpha=0}^{k}\left(\frac{1}{C_\alpha}\right)}$, with variance $b^2 \Big/ \sum_{\alpha=0}^{k}\left(\frac{1}{C_\alpha}\right)$. The best linear invariant

estimator of b based on any given design is then

$$\frac{\sum_{\alpha=0}^{k} b_\alpha^*/c_\alpha}{\sum_{\alpha=0}^{k}\left(\frac{1}{c_\alpha}\right)\left(1 + 1\Big/\sum_{\alpha=0}^{k}\left(\frac{1}{c_\alpha}\right)\right)} = \frac{\sum_{\alpha=0}^{k} b_\alpha^*/c_\alpha}{1 + \sum_{\alpha=0}^{k}\left(\frac{1}{c_\alpha}\right)}$$ with mean squared error given by

$b^2\Big/\left(1 + \sum_{\alpha=0}^{k}\left(\frac{1}{c_\alpha}\right)\right)$. For the traditional design the variance of the best linear unbiased estimator of b is $0.142\ b^2$ and the mean squared error of the best linear invariant estimator is $0.125\ b^2$. For the derived design, the two corresponding values for the expected squared deviation for estimators of b are based on those obtained at only the four stress levels for which $n_\alpha$, $\alpha = 0, 1, \ldots, k$, is greater than 1. They are equal to $0.141\ b^2$ and $0.124\ b^2$, respectively.

The uniformly minimum-variance unbiased estimator of u at $\sigma = \frac{1}{\eta}$ under either model is $u^*(\eta) = \sum_{\alpha=0}^{k} L_\alpha(\eta)\ u^*\left(\frac{1}{\sigma_\alpha}\right)$, where for any $n_\alpha$ equal to 1, $u^*\left(\frac{1}{\sigma_\alpha}\right)$ will be taken to be $X_{1,\alpha} - E(Y_{1,\alpha}) \sum_{i=0}^{k} a_{i,\alpha}\, b_i^*$, with

$$a_{i,\alpha} = \frac{L_\alpha(\eta)E(Y_{1,\alpha})/c_i + \sum_{\substack{j=1 \\ j\neq i}}^{k}\left[L_i(\eta)B_i/(c_ic_j) - L_j(\eta)\ B_j/(c_ic_j)\right]}{L_\alpha(\eta)E(Y_{1,\alpha})\ \sum_{j=1}^{k}(1/c_j)}, \qquad \alpha = 0, 1, \ldots, k.$$

Here, the number of coefficients of the form $a_{1,\alpha}$ at the $\alpha^{th}$ level is equal to 1 minus the number of testing levels for which $n_\alpha$ is equal to 1; and $E(Y_{1,\alpha})$ is the negative of Euler's constant, or approximately $-.577216, \alpha = 0, 1, \ldots, k$. For the traditional model the variance of $u^*(\Pi)$ is 2936 $b^2$, and for the derived model it is 1248 $b^2$. The amount of improvement obtained by using the best linear invariant estimator of $u(\Pi)$ rather than $u^*(\Pi)$ is trivial in both of these cases.

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## APPENDIX B

## EXACT THREE-ORDER-STATISTIC CONFIDENCE BOUNDS ON RELIABILITY PARAMETERS UNDER WEIBULL ASSUMPTIONS

### INTRODUCTION

Assume that a random sample of $n$ items is subjected to a life test until $m$ (with $2 \leq m \leq n \leq 25$) of the items have failed. Assume further that the random variable $T$ which represents an observable failure time from the population from which the sample of failure times is selected has the two-parameter Weibull density given by

$$\lim_{\Delta t \to 0} \frac{P[t < T \leq t + \Delta t]}{\Delta t} = f_{\delta,b}(t) = \begin{cases} (1/\delta b)(t/\delta)^{(1/b)-1} \exp\{-(t/\delta)^{1/b}\}, & t > 0 \\ 0, \text{ otherwise} \end{cases}$$

with $\delta$ and $b$ both positive. (1)

There are two related problems which are among the most important associated with the life-testing situation. First, one may suppose that a required life $t_g$ is specified and a lower confidence bound, based on the censored sample, is sought for reliability or the survival proportion, $R(t_g) = \exp\left\{-(t_g/\delta)^{1/b}\right\}$, at time $t_g$. For the second and alternative problem a survival proportion $R$ is given. Then a lower confidence bound is sought for $t_R = \delta(\log(1/R))^b$, the reliable life corresponding to $R$ or, equivalently, for $x_R = u + b \log \log (1/R)$, the log reliable life, where $u$ is equal to $\log \delta$ and all logarithms are natural. The

parameter u is a location parameter of the distribution of $X = \log T$, and b is a scale parameter of this distribution.[1] The bounds for $t_R$ and $x_R$ are often referred to as lower tolerance limits, and are confidence bounds for the 100(1-R) percent points of the distributions of T and of X, respectively, where R is the specified proportion between 0 and 1.

A method of obtaining confidence bounds on reliability parameters widely used for the Weibull distribution as well as for many others is as follows. Consider the observed ordered failure times $T_1 \leq T_2 \leq \ldots \leq T_m$. For any $T_k$, a lower confidence bound on $R(T_k)$ at level $1 - \alpha$ can be obtained from $v_\alpha(n-k+1, k)$, the $100\alpha$ percent point of the beta distribution with n-k+1 and k degrees of freedom (see Kao and Goode [1]). This bound serves also as a conservative $(1 - \alpha)$-level bound for $R(t_g)$, for any $t_g < T_k$. Similarly, if $v_\alpha(n-k+1, k)$ is greater than R for some k, a lower confidence bound on $t_R$, with level at least $1 - \alpha$, is given by the $k^{th}$ observed failure time $T_k$.

This method, however, makes no use of information concerning the form of the underlying distribution of the populations sampled. It is in fact a nonparametric method requiring only the assumption that the unordered observations are independent and identically and continuously distributed. It seems likely that one could obtain improved bounds on the reliability parameters by assuming (correctly) that sampling is from the two-parameter Weibull distribution. In particular, one would ex-

1 The distribution of the random variable X is the extreme-value distribution of smallest values.

pect a possible improvement when the required life $t_g$ is small compared with the sample failure time which immediately follows it, or in the case in which a tolerance limit is desired, when $v_\alpha$ greatly exceeds R. If R is larger than any $v_\alpha$, the nonparametric method is, in fact, completely inapplicable. In the following discussion, other methods for obtaining bounds are derived.

## TESTS AND CONFIDENCE SETS FOR THE PARAMETER b

### TESTS AND CONFIDENCE SETS BASED ON ALL THE ORDER STATISTICS

The two parameters, reliability and reliable life, for which confidence bounds are sought are

$$R(t_g) = \exp\left\{-(t_g/\delta)^{1/b}\right\} \text{ and } x_R = u + b \log\log(1/R) \text{ or}$$

$t_R = \delta(\log(1/R))^b$, respectively. Thus, both are simply functions of the parameters $\delta$ (or u) and b. Therefore, bounds on the reliability parameters can be determined from distribution percentage points corresponding to appropriate functions of estimates of $\delta$ (or u) and b.

First consider estimates of b (functions of $T_1 \le T_2 \le \ldots \le T_m$) and confidence bounds, based on these estimates, which can be placed on b alone. The problem of obtaining such bounds, or tests from which such bounds are derived, is invariant under a change in scale in the time space or a change of location in log time space. A maximal invariant under this transformation is $Z = (Z_2, Z_3, \ldots, Z_n) = (T_2/T_1, T_3/T_1, \ldots,$

$T_m/T_1)$ or equivalently, $\log Z = [\log (T_2/T_1), \log (T_3/T_1), \ldots, \log (T_m/T_1)] = (X_2-X_1, X_3-X_1, \ldots, X_m-X_1)$. Hence, tests concerning $b$ and confidence bounds on $b$ will be independent of $\delta$ or $u$ if and only if they depend on $T_1, T_2, \ldots, T_m$ or $X_1, X_2, \ldots, X_m$[1] only through $Z$.

The joint probability density function of $T_1 \le T_2 \le \ldots \le T_m$ is given by

$$f_{\delta,b}(t_1, t_2, \ldots, t_m) = \frac{n!}{(n-m)!}\left(\frac{1}{b\delta}\right)^m \prod_{i=1}^{m}\left(\frac{t_i}{\delta}\right)^{(1/b)-1} \exp\left\{-\left[\sum_{i=1}^{m-1}\left(\frac{t_i}{\delta}\right)^{1/b} + (n-m+1)\left(\frac{t_m}{\delta}\right)^{1/b}\right]\right\}, \quad 0 \le t_1 \le t_2 \ldots \le t_m \ldots \le t_n, \tag{2}$$

so that the joint density of $Z_2 \le Z_3 \le \ldots \le Z_m$ is

$$f_b(z_2, z_3, \ldots, z_m) = \frac{n!(m-1)!}{(n-m)!}\left(\frac{1}{b}\right)^{m-1} \frac{\prod_{i=2}^{m} z_i^{(1/b)-1}}{\left[1+\sum_{i=2}^{m-1} z_i^{1/b} + (n-m+1)z_m^{1/b}\right]^m}, \quad 1 \le z_2 \le z_3 \le \ldots \le z_m . \tag{3}$$

Consider any invariant test $\Phi = \Phi(Z)$ of H: $b \ge b_o$ versus K: $b < b_o$ which accepts H when $\Phi = 0$ and rejects H when $\Phi = 1$. One chooses a significance level $\alpha$, $0 < \alpha < 1$, and looks for an invariant test which rejects H with probability $\le \alpha$ when $b \ge b_o$ and which, in some sense, maximizes

[1] $X_1, X_2, \ldots, X_m$ are order statistics from the extreme-value distribution of smallest values, the distribution of the random variate $X = \log T$.

the power $\beta_\Phi(b)$ (or probability of rejecting H) when $b < b_o$. The boundary of the acceptance region of a uniformly most powerful invariant test $\Phi_o$ (most powerful against all alternative hypotheses) yields a uniformly most accurate invariant upper bound $\bar{b}_{min}$ for b. Thus, $P_b\{b \le \bar{b}_{min}\} \ge 1-\alpha$ for all b, and furthermore, for any $b' > b$, $P_b\{\bar{b}_{min} \ge b'\}$ is minimum among tests based on Z. It can be demonstrated that the Neyman-Pearson Fundamental Lemma (Lehmann [2]) applied to (3) for a given n and $m > 2$ and given simple hypothesis, $b = b_o$, will yield a most powerful invariant test which varies accordingly as the simple alternative $b = b_1$ changes. Thus, there is no test of H which is uniformly most powerful among invariant tests for all alternatives.

One might then naturally consider basing tests and confidence sets on estimators of b which have certain optimality properties in terms of their ability to estimate. From [3], [4] and [5] it might be concluded that likely candidates for such bases for tests, etc., would be the best linear invariant estimator and the maximum-likelihood estimator. Unfortunately, however, the derivations of the distributions of these estimators of b, u, and $x_R = u + b \log \log(1/R)$ for $m > 2$ are very difficult, if not impossible, to obtain except by simulation methods.[1]

[1] Tables from which one may obtain confidence bounds on $R(t_g)$ or on $t_R$ have been computed by Johns and Lieberman [6] by simulating the distribution of a function of approximations to the best linear invariant estimates of u and b. These tables can be used for n equal to 10, 15, 20, 30, 50, and 100, and 4 values of m for each n.

A locally most powerful invariant test depends upon the maximum-likelihood estimator of b.

Thus, one is faced with a situation in which statistics which may possibly be optimum in terms of their ability to estimate have not provided an analytical basis for exact tests concerning, or exact confidence bounds on, the parameters they are estimating.

One is thus led to consider estimators whose distributions can be determined exactly and then to compare the properties of these estimators with those of the best linear invariant estimators, the expected loss of which is known or can be calculated for $n \le 25$. We first perform the following derivation proceeding from (3). Let $V_1 = 1$, $V_j = Z_j^{1/b}$, $j = 2, 3, \ldots, m$. Then,

$$f(v_2, v_3, \ldots, v_m) = \frac{n!(m-1)!}{(n-m)!} \frac{1}{\left[n + \sum_{i=2}^{m} (n-i+1)(v_i - v_{i-1})\right]^m}, \qquad (4)$$

$$1 = v_1 \le v_2 \le \ldots \le v_m.$$

Thus, if

$$F = \sum_{i=2}^{m} \frac{n-i+1}{n(m-1)} (v_i - v_{i-1}) = \sum_{i=2}^{m} \frac{n-i+1}{n(m-1)} \left(Z_i^{1/b} - Z_{i-1}^{1/b}\right)$$

then

$$f(F) = \frac{(m-1)^m F^{m-2}}{[1+(m-1)F]^m}, \quad F \ge 0, \qquad (5)$$

or

$$F^{o} = \sum_{i=2}^{m} \frac{n-i+1}{n(m-1)} \left( Z_{i}^{1/b_{o}} - Z_{i-1}^{1/b_{o}} \right) \text{ is distributed as } F[2(m-1),2], \text{ Snedecor's}$$

$F$ with $2(m-1)$ and 2 degrees of freedom, when $b = b_{o}$. A test $\phi_{A}$, at level $\alpha$, based on the statistic $F^{o}$ will therefore reject H: $b \geq b_{o}$ when

$\sum_{i=2}^{m} \frac{n-i+1}{n(m-1)} \left( Z_{i}^{1/b_{o}} - Z_{i-1}^{1/b_{o}} \right)$ is less than $F_{\alpha}[2(m-1),2]$, the $100\alpha$ percent point of the $F$ distribution with the appropriate number of degrees of freedom. The value of $F_{\alpha}[2(m-1),2]$ as a function of $\alpha$ and $m$ can be calculated by letting

$$W^{o} = [1+(m-1)F^{o}]^{-1} = n \left[ 1 + \sum_{i=2}^{m-1} Z_{i}^{1/b_{o}} + (n-m+1) Z_{m}^{1/b_{o}} \right]^{-1}$$

and finding the distribution of $W^{o}$ when $b = b_{o}$. From (5) one obtains

$$f(w^{o}) = (m-1)(1-w^{o})^{m-2}, \quad 0 \leq w^{o} \leq 1,$$

and

$$P\left[F^{o} \leq F_{\alpha}[2(m-1),2]\right] = P\left[W^{o} \geq \bar{W}^{o}\right] = (1-\bar{W}^{o})^{m-1}. \qquad (6)$$

If (6) is set equal to $\alpha$, then $\bar{W}^{o}$ is seen to be $1 - \alpha^{1/(m-1)}$ and $F_{\alpha}[2(m-1),2]$ is equal to $\frac{1}{m-1} \frac{\alpha^{1/(m-1)}}{1-\alpha^{1/(m-1)}}$.

An upper confidence bound at level $1-\alpha$ for $b$ may be obtained by setting

$F = \frac{\sum_{i=2}^{m} (n-i+1)(Z_{i}^{1/b} - Z_{i-1}^{1/b})}{n(m-1)}$ equal to $F_{\alpha}[2(m-1),2]$ and solving for $b$.

One then uses some good estimate of $b$ as a first approximation and Newton-Raphson iteration procedures until $F-F_\alpha[2(m-1),2]$ is sufficiently small.

TESTS AND CONFIDENCE SETS BASED ON TWO ORDER STATISTICS

Next it will be demonstrated that, for $m$ sufficiently small, a family of confidence bounds based on $Z_m = T_m/T_1$ alone is preferable to that based on the test $\Phi_A$ (just derived). To obtain the density function for $Z_m$, one may successively integrate out $Z_2^{1/b}$, $Z_3^{1/b}$, ..., $Z_{m-1}^{1/b}$ from (3). Thus,

$$f(z_m) = \frac{n!}{(n-m)!(m-2)!} \sum_{k=0}^{m-2} \left(\frac{-1}{b}\right)^k \binom{m-2}{k} \frac{z_m^{(1/b)-1}}{\left[(m-k-1)+(n-m-k+1)z_m^{1/b}\right]^2}, \quad (7)$$

$$1 \leq z_m < \infty .$$

Now, let $V = Z_m^{1/b} \geq 1$. A test $\Phi_1$ based on $Z_m$, therefore, rejects $H: b \geq b_o$ whenever $V_o = Z_m^{1/b_o}$ is less than $C_1(\alpha)$(or $\log Z_m$ is less than $b_o \log C_1(\alpha)$), with $C_1(\alpha)$ determined from the expression

$$\frac{n!}{(n-m)!(m-2)!} \sum_{k=0}^{m-2} (-1)^k \binom{m-2}{k} \frac{1/(n-m+k+1)}{[(m-k-1)+(n-m+k+1)C_1(\alpha)]} = 1 - \alpha. \quad (8)$$

An upper confidence bound $\bar{b}_1(\alpha)$ for $b$ will then be given by $\bar{b}_1(\alpha) = \log Z_m/\log C_1(\alpha) = \log Z_m/LC_1(\alpha)$. In order to ascertain whether there is a unique root of the equation (8) which is greater than unity, let $D_1(\alpha) = C_1(\alpha)-1$ and reject $H$ when $Z_m^{1/b_o} -1 < D_1(\alpha)$.

Then (8) may be expressed as a function of $D_1(\alpha)$

$$\frac{n!}{(n-m)!(m-2)!}\sum_{k=0}^{m-2}(-1)^k\binom{m-2}{k}\frac{1/(n-m+k+1)}{[n+(n-m+k+1)D_1(\alpha)]} = 1-\alpha$$

or as a polynomial $g(D_1(\alpha))$ given by

$$(1-\alpha)\,D_1(\alpha)^{m-1}\prod_{k=1}^{m-1}(n-m+k) - \alpha\left\{D_1(\alpha)^{m-2}\sum_{i=1}^{\binom{m-1}{1}} n \prod_{\substack{k=1\\k\neq i}}^{m-1}(n-m+k)\right.$$

$$\left. + \cdots + D_1(\alpha)\sum_{i=1}^{\binom{m-1}{m-2}} n^{m-2}(n-m+i) + n^{m-1}\right\} = 0 \;. \qquad (9)$$

For $0 < \alpha < 1$, the coefficient of the first term of the polynomial is positive and each of the remaining $m - 1$ terms is negative. Thus by Descartes' Rule of Signs [7], there is no more than one positive root. Also, since $g(0) < 0$ and $g(D_1(\alpha))$, for $D_1(\alpha)$ very large, is positive, there is <u>at least</u> one positive root. Therefore, there is a unique $D_1(\alpha) > 0$ and a unique $C_1(\alpha) = D_1(\alpha) + 1 > 1$.

It can be easily demonstrated from equation (9) that when $n$ is sufficiently large compared with $m$, $D_1(\alpha) = \dfrac{\alpha^{1/(m-1)}}{1 - \alpha^{1/(m-1)}} = (m-1)F_\alpha[2(m-1),2]$ and from Table IX in [8] it may be corroborated that, for $m \leq 20$, $LC_1(\alpha) = \log\,(1+D_1(\alpha))$ is approximately given by

$$\log\left[n\sum_{i=2}^{m}\frac{1}{n-i+1}\,\frac{\alpha^{1/(m-1)}}{(m-1)(1-\alpha^{1/(m-1)})}+1\right] \tag{10}$$

or

$$\log\left[n\sum_{i=2}^{m}\frac{1}{n-i+1}\,F_{\alpha}[2(m-1),2]+1\right].$$

This result has been demonstrated empirically using the fact that (10) was used as a first approximation in the Newton-Raphson iterative procedure employed to generate $LC_1(\alpha)$ and the amount of deviation from this first guess is shown in the machine computation output to be small. Thus, except for large $m$, the test $\Phi_1$ rejects $H\colon b \geq b_o$ approximately when

$$\log Z_m \Big/ \left\{b_o \log\left[n\sum_{i=2}^{m}\frac{1}{n-i+1}\,F_{\alpha}[2(m-1),2]+1\right]\right\} < 1 .$$

COMPARISON OF BOUNDS BASED ON $\Phi_1$ AND $\Phi_A$

Very roughly, with probability $1-\gamma$ the statistic $\log Z_m$ is greater than $b\log\left[n\sum_{i=2}^{r}\frac{1}{n-i+1}F\left[2_{\gamma}(m-1),2\right]+1\right]$ and with probability $\gamma$ is less than this value, as long as $m$ is not large. Next, the statistic

$$\sum_{i=2}^{m}(n-i+1)\left(\frac{Z_i^{1/b_o}-Z_{i-1}^{1/b_o}}{n(m-1)}\right)=\frac{1}{n}\left[\sum_{i=2}^{m}\left(\frac{Z_i^{1/b_o}-1}{m-1}\right)+(n-m+1)\left(\frac{Z_m^{1/b_o}-1}{m-1}\right)\right],\quad Z_1=1,$$

upon which the test $\Phi_A$ is based can be considered to be $\frac{n-1}{n}\left(\frac{Z_{\mu}^{1/b_o}-1}{m-1}\right)$ for some $\mu$ with $2 \leq j \leq \mu \leq k \leq m$, $j$, $k$, and $m$ integers and

$Z_j \le Z_\mu \le Z_k$. (When $m = 2$, $\mu = 2$, and $\Phi_A$ is equivalent to $\Phi_1$). Then, from (5) H is rejected on the basis of $\Phi_A$ when

$$\log Z_\mu / b_0 \log \left[\frac{n(m-1)}{n-1} F_\alpha[2(m-1),2] + 1\right] < 1 ,$$

with the $\gamma^{th}$ percentage point of the distribution of $\log Z_\mu$ equal to

$$b \log \left[\frac{n(m-1)}{n-1} F_\gamma[2(m-1),2] + 1\right] .$$

For either of the two $(1-\alpha)$-level confidence bounds based on $\Phi_1$ and $\Phi_A$ then, with approximate probability $\gamma$ for small $m$, the upper bound $\bar{b}$ is less than

$$\frac{b \log(F_\gamma[2(m-1),2]\, x+1)}{\log(F_\alpha[2(m-1),2]\, x+1)} , \qquad (11)$$

where

$$x = \frac{n}{n-1}(m-1), \qquad x = n \sum_{i=2}^{m} \frac{1}{n-i+1} \ge n \sum_{i=2}^{m} \frac{1}{n-1} = \frac{n(m-1)}{n-1} > 1.$$

(When $m=2$, the two expressions for $x$ are equal.) At confidence level $1-\alpha$, therefore, the test ($\Phi_1$ or $\Phi_A$) which yields the larger value of (11) for $\gamma < \alpha$ $(1-\gamma > 1-\alpha)$ may be considered superior since it is therefore the more accurate bound. Accuracy of the bound is in the sense of the definition of a most accurate bound as found in [2] and on page 5.

If the expression

$$\frac{\log(ax+1)}{\log(cx+1)} , \; x > 0, \; 0 < a < c , \qquad (12)$$

is differentiated, one obtains

$$\frac{-\log(ax+1)\frac{c}{cx+1} + \log(cx+1)\,\frac{a}{ax+1}}{\log^2(cx+1)} = \frac{-\frac{ax+1}{ax}\log(ax+1) + \frac{cx+1}{cx}\log(cx+1)}{\frac{cx+1}{c}\cdot\frac{ax+1}{a}\cdot\frac{1}{x}\log^2(cx+1)} \tag{13}$$

Now consider $\frac{y+1}{y}\log(y+1)$ whose derivative is $\frac{1}{y} + \log(1+y)(-1/y^2)$.

Since $y > \log(1+y)$, this derivative is always positive and $\frac{y+1}{y}\log(1+y)$ is increasing in $y$.[1] Therefore, since $a < c$, the right hand term in the numerator of (13) is larger than the left, (13) is always positive, and (12) is increasing in $x$. Hence the larger the value of $x$, $\frac{n(m-1)}{n-1}$ or $n\sum_{i=2}^{m}\frac{1}{n-i+1}$, the more desirable the associated confidence bound.

Then, since $n\sum_{i=2}^{m}\frac{1}{n-i+1} > \frac{n(m-1)}{n-1}$ for $m > 2$, an upper confidence bound based on $\Phi_1$ is more accurate than one based on $\Phi_A$ for small $m$.

UNBIASEDNESS OF $\Phi_1$

The test $\Phi_1(Z_m)$ can be shown to be unbiased in the following manner. The power function $\beta_{\Phi_1}(b)$ of $\Phi_1$ can be expressed as a function of $C_1(\alpha)$,

$$\beta_{\Phi_1}(b) = E_b(\Phi_1) = 1 - \frac{n!}{(n-m)!(m-2)!}\sum_{k=0}^{m-2}(-1)^k\binom{m-2}{k}\frac{1}{[(m-k-1)+(n-m+k+1)\left(C_1(\alpha)\right)^{b_0/b}]} \tag{13}$$

[1] The form of the right side of (13) and this step of the proof were suggested by Prof. Wayne E. Smith of the University of California at Los Angeles.

and the derivative $\beta'_{\Phi_o}(b)$ thus has the form

$$\beta'_{\Phi_1}(b) = \frac{-n!}{(n-m)!(m-2)!}\left(C_1(\alpha)\right)^{b_o/b}\log\left(C_1(\alpha)\right)^{b_o}\sum_{k=0}^{m-2}\frac{(-1)^k}{b^2}\binom{m-2}{k}\frac{1}{\left[(m-k-1)+(n-m+k+1)\left(C_1(\alpha)\right)^{b_o/b}\right]^2}. \quad (14)$$

Then from (5) and (7),

$$\beta'_{\Phi_1}(b) = -\frac{1}{b^2}\left(C_1(\alpha)\right)^{b_o/b}\log\left(C_1(\alpha)\right)^{b_o}\cdot f(v)\Big|_{v=\left(C_1(\alpha)\right)^{b_o/b}} < 0$$

since $C_1(\alpha) > 1$. Hence, $\beta_{\Phi_1}(b)$ is monotonically decreasing in $b$. For $\Phi_1$ the two criteria for unbiasedness are satisfied, that is

$$\beta_{\Phi_1}(b) \le \alpha \text{ for } b \ge b_o$$

$$\beta_{\Phi_1}(b) > \alpha \text{ for } b < b_o.$$

Satisfaction of the first of these criteria insures that the test is of size $\alpha$ or that $\sup \beta_{\Phi_1}(b) = \alpha$ for $b \ge b_o$ since $\beta_{\Phi_1}(b_o) = \alpha$ .

## INDUCED TESTS AND CONFIDENCE SETS FOR RELIABILITY AND RELIABLE LIFE

### INDUCED TESTS AND CONFIDENCE SETS FOR RELIABILITY BASED ON ALL THE ORDER STATISTICS

Suppose that the matter of concern is the problem of testing that $R(t_g) = \exp\left[-(t_g/\delta)^{1/b}\right]$, the proportion of the population surviving

at a specified time $t_g$, is less than some given $R_o$. Let $\xi = \delta^{1/b} = t_g^{1/b}/\log[1/R(t_g)]$. The distribution of

$$\frac{2m\hat{\xi}}{\xi} = \frac{2\left(\sum_{i=1}^{m-1} T_i^{1/b} + (n-m+1)T_m^{1/b}\right)}{\xi}$$

is $\chi^2_{2m}$; and when $b$ is known (see [10]) a uniformly most powerful $\alpha'$-level test $\phi_2$ of $H'$: $\delta^{1/b} \le \delta_o^{1/b}$, or equivalently, since $R(t_g) = \exp\left[-(t_g/\delta_o)^{1/b}\right]$, $H'$: $R(t_g) \le R_o = \exp\left[-(t_g/\delta_o)^{1/b}\right]$ versus $K$: $R(t_g) > R_o$ is: Reject $H'$ when $2m\hat{\xi}/t_g$ is greater than $\chi^2_{2m}(1-\alpha)/\log 1/R_o$ (accept otherwise). Therefore $H$ is accepted at level $\alpha'$ if

$$2\left[\sum_{i=1}^{m-1} \frac{T_i}{t_g}^{1/b} + (n-m+1) \frac{T_m}{t_g}^{1/b}\right] \le \chi^2_{2m}(1-\alpha')/\log(1/R_o) \; . \qquad (15)$$

The left side of (15) increases monotonically as $b$ decreases if $t_g < T_1^{T_1^{1/b}/\Delta} \cdot T_2^{T_2^{1/b}/\Delta} \cdot \ldots \cdot T_m^{(n-m+1)T_m^{1/b}/\Delta}$, with $\Delta = T_1^{1/b} + T_2^{1/b} + \ldots + (n-m+1)T_m^{1/b}$, and certainly if $t_g \le (T_1 \cdot T_2 \cdot \ldots \cdot T_m)^{1/m}$. Thus for $t_g$ sufficiently small, if $H'$ is accepted under the assumption that $b = b_o$ when the value of $b_o$ is inserted for $b$ in (15), then it would surely be accepted if $b$ were greater than $b_o$ and the true value of $b$ were somehow to be substituted for the symbol $b$ in (15).

Hence one may form an induced test $\phi^*$ of $H^*$ ($H$: $b \geq b_o$ is true and $H'$ is true) versus $K^*$ ($H^*$ is not true) which uses the test $\phi_2$ with $b = b_o$ in (15) to test $H'$ only if the hypothesis $H$ is accepted on the basis of the test $\phi_1$. Now, consider

$$m\hat{\xi} = \left[\sum_{i=1}^{m-1} T_i^{1/b} + (n-m+1)T_m^{1/b}\right] = \sum_{i=1}^{m} (n-i+1)\left(T_i^{1/b} - T_{i-1}^{1/b}\right), \quad (T_o = 0).$$

The joint density of $W_i = T_i^{1/b}$, $i = 1, 2, \ldots, m$, is

$$f(w_1, w_2, \ldots, w_m) = \left[\prod_{i=1}^{m} (n-i+1)/\delta^{1/b}\right] \exp\left\{-\frac{1}{\delta^{1/b}}\left[n\, w_1 + (n-1)(w_2 - w_1) + \ldots + (n-m+1)(w_m - w_{m-1})\right]\right\}, \quad \begin{matrix} 0 \leq w_1 < \infty, \\ w_1 \leq w_2 < \infty, \\ w_{m-1} \leq w_m < \infty. \end{matrix}$$

Thus the joint density of $S_i = (n-i+1)(W_i - W_{i-1}) = (n-i+1)(T_i^{1/b} - T_{i-1}^{1/b})$, $T_o = 0$, $i = 1, 2, \ldots, m$, is

$$f(s_1, s_2, \ldots, s_m) = \left(\frac{1}{\delta^{1/b}}\right)^m \exp\left[-\left(\frac{1}{\delta^{1/b}} \sum_{i=1}^{m} s_i\right)\right],$$

$$0 \leq s_i < \infty, \; i=1, 2, \ldots, m$$

and each $S_i = (n-i+1)\left(T_i^{1/b} - T_{i-1}^{1/b}\right)$, $i = 1, 2, \ldots, m$, has an independent gamma distribution with scale parameter $\delta^{1/b}$. Therefore, by a result of Lukacs in [9], $m\hat{\xi} = \sum_{i=1}^{r} (n-i+1)(T_i^{1/b} - T_{i-1}^{1/b})$ is independent of any of m-1 linear combinations of

$$1, \; \frac{(n-1)[T_2^{1/b} - T_1^{1/b}]}{nT_1^{1/b}}, \; \frac{(n-2)[T_3^{1/b} - T_2^{1/b}]}{nT_1^{1/b}}, \; \ldots \text{ and } \frac{(n-m+1)[T_m^{1/b} - T_{m-1}^{1/b}]}{nT_1^{1/b}}$$

which are equal to

$1, \frac{n-1}{n}(Z_2^{1/b}-1), \frac{n-2}{n}(Z_3^{1/b}-Z_2^{1/b}), \ldots,$ and $\frac{(n-m+1)}{n}(Z_m^{1/b}-Z_{m-1}^{1/b})$ .

Therefore, $m\hat{\xi}$ is independent of $Z_2^{1/b}$, $Z_3^{1/b}$, ..., and $Z_m^{1/b}$ and hence of $Z_m$. Darroch and Silvey [11] show that when the two tests which form the induced test are similar ($\beta_{\phi_1}(b_o) = \alpha$, $\beta_{\phi_2}(\xi_o) = \alpha'$), of size $\alpha$ and $\alpha'$, respectively, and are based on independent test statistics,[1] then the induced test is similar and of size $1-(1-\alpha)(1-\alpha')$. Since unbiasedness of a test with a continuous power function implies both that the test is similar and that the size is equal to the significance level, these conditions are satisfied for both $\phi_1$ and $\phi_2$ (see [10]). Thus $\phi^*$ is similar and of size $1-(1-\alpha)(1-\alpha')$.

Now, just as the tests $\phi_1$ and $\phi_2$ have been combined to form the induced test $\phi^*$, the boundaries of the acceptance regions of the two tests may be combined to form joint confidence bounds for $b$ and $R(t_g)$ at confidence level $(1-\alpha)(1-\alpha')$. This is possible because $\phi^*$ is similar and of size $1-(1-\alpha)(1-\alpha')$. From the acceptance region of the test $\phi_2$ of $H'$: $R(t_g) \le R_o$ versus $K'$: $R(t_g) > R_o$, with probability $1-\alpha'$,

[1] The expression $\left[\sum_{i=1}^{m-1}(t_i)^{1/b} + (n-m+1)T_m^{1/b}\right]$ is considered a statistic even though it contains the parameter $b$ since the value of $b$ here is assumed known.

$$R(t_g) \geq \exp\left\{-\frac{\chi^2_{2m}(1-\alpha')}{2\left[(T_1/T_g)^{1/b}+(T_2/t_g)^{1/b}+\ldots+(n-m+1)(T_m/t_g)^{1/b}\right]}\right\}, \quad (16)$$

where the right side of (16) is monotonically decreasing in b as long as $t_g < T_1^{T_1^{1/b}/\Delta} \cdot T_2^{T_2^{1/b}/\Delta} \cdot \ldots \cdot T_m^{(n-m+1)T_m^{1/b}/\Delta}$ with

$\Delta = T_1^{1/b} + T_2^{1/b} + \ldots + (n-m+1)T_m^{1/b}$, and certainly if $t_g \leq (T_1 \cdot T_2 \cdot \ldots \cdot T_m)^{1/m}$.

Thus, if $\bar{b} = \bar{b}_1(\alpha)$ (the boundary of the acceptance region of $\xi_1$) is equal to $\log Z_m/LC_1(\alpha)$,

$$R(t_g) \geq \exp\left\{-\frac{\chi^2_{2m}(1-\alpha)}{2\left[(T_1/t_g)^{1/\bar{b}}+(T_2/t_g)^{1/\bar{b}}+\ldots+(n-m+1)(T_m/t_g)^{1/\bar{b}}\right]}\right\}$$

(for $t_g$ sufficiently small) is true with probability at least $(1-\alpha)(1-\alpha')$.[1]

INDUCED TESTS AND CONFIDENCE SETS BASED ON TWO OR THREE ORDER STATISTICS

Darroch and Silvey [11] show that the two statistics upon which an induced test is based need not be independent. As long as the tests are similar and of size $\alpha$ and $\alpha'$, then the induced test will be similar and of size $\alpha''$ with $\max(\alpha, \alpha') \leq \alpha'' \leq \alpha + \alpha'$. Moreover, even though $\hat{\xi}$ gives a uniformly most accurate confidence bound and uniformly most powerful tests for functions of $\xi = \delta^{1/b}$ when b is known, it does not necessarily follow that this situation holds when b is not known. We therefore consider the following one-order-statistic

[1] This inequality containing $\delta$ and the inequality $b \leq \bar{b}$ are jointly true with probability $(1-\alpha)(1-\alpha')$ exactly.

bound for $\xi$ which allows for hand computation of test statistics and bounds for $R(t_g)$ with $t_g$ specified and also for $t_R$, when alternatively a survival proportion $R$ is specified. When $b$ is known, a test $\Phi_3$ of $H': \xi \le \xi_0 = \delta_0^{1/b}$ versus $K': \xi > \xi_0$ can be obtained from any one order statistic, say $T_\nu$, as shown in [12].[1] If one uses such a test, $H'$ is rejected at level $\alpha'$ when

$$T_\nu^{1/b} > \xi_0/B_\nu(\alpha') ,$$

where $B_\nu(\alpha')$ is equal to $-\frac{1}{\log v_{\alpha'}(n-\nu+1,\nu)}$ and $v_{\alpha'}(n-\nu+1,\nu)$ is the $100\alpha'$ percent point of the beta distribution with $n-\nu+1$ and $\nu$ degrees of freedom (see [13]). Then, since $R(t_g) = \exp\left[-(t_g/\delta)^{1/b}\right]$, $t_g > 0$, if $t_g$ is specified, the hypothesis $H'$ can be expressed as $R(t_g) \le R_{\delta_0,b}(t_g) = R_0$ and will be rejected when

$$(T_\nu/t_g)^{1/b} > \left[B_\nu(\alpha') \log (1/R_0)\right]^{-1} \tag{17}$$

Alternatively, if a survival proportion $R$ is specified, $H'$ may be expressed by $t_R \ge t_R^0 = \delta_0(\log(1/R))^b$ and rejected when

$$(B_\nu(\alpha') \log (1/R))^b > t_R^0/T_\nu . \tag{18}$$

The left side of each of the two inequalities (17) and (18) is monotonically decreasing or increasing in $b$ depending upon whether $t_g$ is less than or greater than $T_\nu$ or $B_\nu(\alpha')\log(1/R)$ is less than or greater than 1, respectively. Suppose $t_g < T_\nu$ or $B_\nu(\alpha')\log(1/R) < 1$ and $H'$ is accepted

[1] In [12] $T_\nu$ is actually the order statistic yielding smallest expected squared deviation of the bound, based on the acceptance region of $\Phi_3$, from $\xi$.

at significance level $\alpha$ on the basis of $\Phi_3$ and the assumption that b is equal to $b_o$. Suppose also that the hypothesis H: $b \geq b_o$ has been accepted at level $\alpha$ on the basis of the test $\Phi_1$. Then the induced test of $H^*$ (H and $H'$ both true) has size and significance level $\alpha''$, and $\max(\alpha, \alpha') \leq \alpha'' < (\alpha + \alpha')$, as long as the test $\Phi_3$ is similar and of size $\alpha'$. As noted earlier, unbiasedness of a test with continuous power function and significance level $\alpha'$ implies that the test is similar and of size $\alpha'$. Therefore, it is sufficient to demonstrate that $\Phi_3$ is unbiased in order to guarantee that $\alpha''$, the size and the significance level of the induced test have the lower and upper limits $\max(\alpha, \alpha')$ and $\alpha + \alpha'$, respectively. This is shown in the following steps.

First, if $T_\nu$ is the $\nu^{th}$ order statistic of a size n sample from the Weibull distribution, then $(T_\nu)^{1/b}$ is equivalent to the $\nu^{th}$ order statistic $W_\nu$ of a size n sample from the negative exponential distrition with scale parameter $\xi$. Thus, the power function of the test $\Phi_3$ is given by

$$\beta_{\Phi_3}(\xi) = \int_{\xi_o/B_\nu(\alpha')}^{\infty} \frac{n!}{(\nu-1)!(n-\nu)!\xi} e^{-(n-\nu+1)\, w_\nu/\xi} \left[1-e^{\,w_\nu/\xi}\right]^{\nu-1} dw_\nu$$

or

$$\beta_{\Phi_3}(\xi) = \frac{n!}{(\nu-1)!(n-\nu)!} \sum_{k=o}^{\nu-1} (-1)^k \binom{\nu-1}{k} \frac{e^{-(n-\nu+k+1)\, \xi_o/\xi B_\nu(\alpha')}}{(n-\nu+k+1)}$$

Therefore,

$$\beta'_{\phi_3}(\xi) = \frac{n!}{(\nu-1)!(n-\nu)!}\sum_{k=0}^{\nu-1}(-1)^k\binom{\nu-1}{k}\left[\xi_o/\xi^2 B_\nu(\alpha')\right]e^{-\left[(n-\nu+k+1)\xi_o/\xi B_\nu(\alpha')\right]}$$

$$= \left[\xi_o/(\xi\, B_\nu(\alpha'))\right] f(w_\nu)\Big|_{w_\nu=\xi_o/B_\nu(\alpha')}$$

Then, since $\xi$, $\xi_o$, $B_\nu(\alpha')$, and $f(\xi_o/B_\nu(\alpha'))$ are each greater than zero, $\beta'_{\phi_3}(\xi)$ is positive for all $\xi$ and the power function is monotonically increasing in $\xi$. (The power function of the dual test $\phi_3$ of $K'$ versus $H'$ can in like manner be shown to be monotonically decreasing in $\xi$.) Hence, both tests are unbiased, and the size and significance level of the induced test are equal to $\alpha''$.

A confidence bound for $R(t_g)$ or $t_R$ can then be obtained from the boundary of the acceptance region of the test $\phi_3$ with

$$P\left\{R(t_g) \geq \exp\left[-(t_g/T_\nu)^{1/\bar{b}}/B_\nu(\alpha')\right]\right\} \geq 1-\alpha'', \; t_g < T_\nu \tag{19}$$

or

$$P\left\{t_R \geq T_\nu\left[B_\nu(\alpha')\log(1/R)\right]^{\bar{b}}\right\} \geq 1-\alpha'', \; B_\nu(\alpha')\log(1/R) < 1 \tag{20}$$

and $\bar{b} = \bar{b}_1(\alpha)$ the boundary of the acceptance region of $\phi_1$ at confidence level $1-\alpha$. Alternatively, then, one may write (19) as

$$P\left\{\log\left(\log(1/R\,(t_g)\right) \leq LC_1(\alpha)\frac{\log(t_g/T_\nu)}{\log(T_m/T_1)} - \log B_\nu(\alpha')\right\} \geq 1-\alpha'',$$

$$t_g < T_\nu,$$

and (20) as

$$P\left\{\log t_R \ge \log \underline{t_R} = \log T_\nu + \left[\log(T_m/T_1)/LC_1(\alpha)\right]\left[\log(\log(1/R)) + \log B_\nu(\alpha')\right]\right\} \ge 1-\alpha'' . \quad (21)$$

for $\log(\log(1/R)) + \log B_\nu(\alpha') < 0$.

## IMPROVEMENT OF DERIVED TESTS AND CONFIDENCE SETS

### TESTS AND CONFIDENCE SETS CONCERNING b

Now, suppose one is no longer restricted to the use of the order statistics $T_1$ and $T_m$ for obtaining tests and confidence sets for $b$, but instead considers any two order statistics as possible candidates upon which to base these tests and confidence sets. Suppose one uses as a criterion in choosing a confidence bound of the form $(\log T_q - \log T_p)/k_{p,q}$, smallest expected squared deviation of the bound from the parameter $b$ (see Harter [14]). It can easily be shown that there exists a combination of $n$ and $m$ for which the combination of $\log T_1$ and $\log T_m$ does not provide the linear invariant two-order-statistic estimate of $b$ that has smallest mean squared deviation from this parameter. Therefore, for each combination of $n$, $m$, and $\alpha$ considered, a coefficient, $LC_{p,q}(\alpha)$, yielding an upper bound $\bar{b}_{p,q}(\alpha) = (\log T_q - \log T_p)/LC_{p,q}(\alpha)$, $1 \le p < q \le m$ and the mean squared deviation of $\bar{b}_{p,q}(\alpha)$ from $b$ have been calculated using

results obtained in [8]. The combination of p and q yielding the most efficient bound, that is, the one satisfying the specified criterion, is listed in Table B.I along with values of LC $= LC_{p,q}(\alpha)$ from which the bound may be obtained. Also in preparation is an additional table which will list, for selected values of $\alpha$, the ratio of the mean squared errors of the best linear invariant estimators of b based on (1) all m available order statistics and (2) the combination of $X_p$ and $X_q$ yielding the most efficient bound.

Consider the power function $\beta_{\phi_{p,q}}(b)$ of the test $\phi_{p,q}$ of H: $b \geq b_o$ versus K: $b < b_o$ at significance level $\alpha$ which rejects the hypothesis when

$$(\log T_q - \log T_p)/b_o \leq LC_{p,q}(\alpha).$$

Proceeding from (3) it can be shown that if $C_{p,q}(\alpha) = \exp\left(LC_{p,q}(\alpha)\right)$, then $\beta_{\phi_{p,q}}(b)$ has the form

$$\beta_{\phi_{p,q}}(b) = E_b(\phi_{p,q}) = 1 - \frac{n!}{(p-1)!(q-p-1)!(n-q)!}\sum_{k=o}^{p-1}\sum_{j=o}^{q-p-1}(-1)^{k+j}\binom{p-1}{k}\binom{q-p-1}{j}$$

$$\frac{1/(n-q+j+1)}{\left((q-p-j+k)+(n-q+j+1)\left(C_{p,q}(\alpha)\right)^{b_o/b}\right)}$$

Thus, as in the case in which $T_p = T_1$, $T_q = T_m$, and $C_{p,q}(\alpha) = C_1(\alpha)$, it follows that

$$\beta'_{\Phi_{p,q}}(b) = -\frac{1}{b^2}\left(C_{p,q}(\alpha)\right)^{b_o/b}\log\left(C_{p,q}(\alpha)\right)^{b_o} f(s)\Bigg|_{s=\left(C_{p,q}(\alpha)\right)^{b_o/b}}$$

Then, as before, $C_{p,q}(\alpha)$ is greater than 1 (since $p$ is less than $q$), and $\beta'_{\Phi_{p,q}}(b)$ is negative. Therefore the power function is monotonically decreasing in $b$. This insures that the test is unbiased and that an induced test of $H^*$ ($H: b \geq b_o$ is true and $H': \xi \leq \xi_o$ is true) based on $\Phi_{p,q}$ at significance level $\alpha$ and $\Phi_3$ at significance level $\alpha'$ will be similar and of size $\alpha''$ with $\max(\alpha, \alpha') \leq \alpha'' \leq \alpha + \alpha'$ as when $\Phi_{p,q} = \Phi_{1,m} \equiv \Phi_1$. The value of the power function $\beta_{\Phi_{p,q}}(b)$ was calculated for $b/b_o$ equal to 0.25, 0.5 and 0.75 for each combination of $p$ and $q$ corresponding to each combination of $n$, $m$, and $\alpha$ so that the relative general sizes of the power functions of the tests for $b < b_o$ (related to the accuracy of the bound) could be compared with the relative sizes of the mean squared deviations of the bounds from the parameter.

It was found that the test based on the combination of $p$ and $q$ providing the most efficient bound for each combination of $n$, $m$, and $\alpha$ is not in general uniformly most powerful. That is, it is not always most powerful among tests based on two order statistics for all three of the values of $b/b_o$ calculated. In each case, however, the test corresponding to the most efficient bound appears to be if not uniformly most powerful, at least either locally most powerful or most powerful for $b/b_o$ close to zero. Furthermore, whenever a test

associated with a bound which is not the most efficient has higher power than that based on the most efficient combination of $X_p$ and $X_q$ for any value of $b/b_0$ calculated, the power functions of the two tests differ only slightly and the mean squared deviations of the two bounds are very nearly equal.

For example, the computed values of the power functions, for $\alpha = 0.05$, $n = m = 16$, and $b/b_0$ equal to 0.25, 0.50, and 0.75, are for the test based on the most efficient combination $X_3$ and $X_{16}$ equal to 0.999997, 0.890903, and 0.296701, respectively, and for the test based on $X_4$ and $X_{16}$ equal to 0.999999, 0.882135, and 0.283704, respectively. The expected squared deviations of the two corresponding bounds are 0.4289 and 0.4470, respectively.

TESTS AND CONFIDENCE SETS CONCERNING RELIABLE LIFE

It can also be shown that among lower bounds on $x_R$ at confidence level $\geq 1 - \alpha'' \geq 1 - (\alpha + \alpha')$ of the form

$$\log T_\nu + \frac{\log \log (1/R) + B_\nu(\alpha')}{LC_{p,q}(\alpha)} (\log T_q - \log T_p), \; 1 \leq p < q \leq m, \quad (22)$$

(as defined in (21)), the one with smallest expected squared deviation does not necessarily have $T_q = T_m$ and $T_p = T_1$ and $LC_{p,q}(\alpha) = LC_1(\alpha)$. Furthermore, an additional improvement can be made in the following

manner. Consider the identity,

$$X_\nu + \frac{\log\log(1/R) - Y_\nu}{Y_q - Y_p}(X_q - X_p) \equiv u + b \log\log(1/R) = x_R \quad , \tag{23}$$

$$1 \le p < q \le m,$$

where $Y_k = (X_k - u)/b$ and $X_k = \log T_k$ for any k. Hence, if

$\dfrac{\log\log(1/R) - Y_\nu}{Y_q - Y_p} \ge \underline{V}$ with probability exactly equal to $1 - \alpha^*$, then

$$P\left\{x_R \ge X_\nu + \underline{V}(X_q - X_p)\right\} = 1 - \alpha^*.$$

Call $\dfrac{\log\log(1/R) + B_\nu(\alpha')}{LC_1(\alpha)}$ in (21), $\underline{W}$, and call the bound defined by (22), $\underline{x_R}(W)$. From (22) and (23) $\dfrac{\log\log(1/R) - Y_\nu}{Y_q - Y_p} \ge \underline{W}$ with probability greater than or equal to $1 - (\alpha + \alpha')$. Thus, if $1 - \alpha^*$ is equal to $1 - (\alpha + \alpha')$, $\underline{W}$ is less than or equal to $\underline{V}$ and $X_\nu + \underline{W}(X_q - X_p)$ is less than or equal to $X_\nu + \underline{V}(X_q - X_p)$. It follows that for any value $x_R' < x_R$ the probability that $X_\nu + \underline{V}(X_q - X_p)$ is less than $x_R'$ is smaller than or equal to the probability that $X_\nu + \underline{W}(X_q - X_p)$ is less than $x_R'$. The more accurate bound is therefore given by

$$\underline{x_R}(V) = X_\nu + \underline{V}(X_q - X_p).$$

Moreover, since the test associated with the bound $\underline{x_R}(W)$ is of size $\le (\alpha + \alpha') = \alpha^*$, the test associated with the bound $\underline{x_R}(V)$ is also of

size $\leq \alpha^*$, hence of size $\alpha^*$ since $\alpha^*$ is the significance level. Since it is not possible by calculation and examination of the power function to demonstrate that the test associated with the bound $\underline{x_R(V)}$ is unbiased, this is of interest.

It was originally planned that the combination of $p$, $q$, and $\nu$ yielding a most efficient bound for $x_R$ would be determined for each combination of $n$, $m$, $R$, and $\alpha$ considered. Doing this, however, would require iterative calculations of 20,615 percentage points of the distribution of $V$ from the expression (or an expression similar to that given) on page 77 for each combination of $\alpha$ and $R$, if only sample sizes 2, 3, ..., 20 were considered. Since several seconds, at least, appear to be required to perform the numerical integrations involved in calculating each percentage point, it has been decided to cut down on the number of necessary calculations by choosing the three order statistics on which to base the bounds before any values of $\underline{V}$ are computed. For each combination of sample size $n$, censoring number $m$, and specified survival proportion $R$, the estimator of $x_R$ with smallest risk among those of the form $X_\nu + C(X_q - X_p)$, $1 \leq \nu \leq m$, $1 \leq p < q \leq m$, will be selected. The three order statistics specified by this estimation rule will then provide the basis for bounds at all confidence levels for each corresponding combination of $n$, $m$, and $R$. Indications are that very little increase in risk for the bounds will result from this procedure; and the number of percentage points to be calculated for each combination of $\alpha$ and $R$ will be reduced to 171.

OBTAINING BOUNDS ON RELIABILITY FOR SPECIFIED REQUIRED LIFE

Suppose that required life $t_g$ rather than a survival proportion $R$ is specified. Then it will often be possible to use the values of $\underline{V}$ which will be tabulated to obtain an upper confidence bound on $R(t_g)$, the proportion surviving at time $t_g = \exp(x_g)$.

The bounds on $x_R$ or $t_R$, referred to immediately above, depend on the identity

$$x_R = u + b \log \log (1/R) \equiv X_\nu + \frac{\log \log (1/R) - Y_\nu}{Y_q - Y_p} (X_q - X_p). \qquad (24)$$

This can also be written in the form:

$$\log \log (1/R(t_g)) \equiv \frac{X_\nu - u}{b} + \frac{x_g - X_\nu}{X_q - X_p} \; \frac{X_q - X_p}{b}$$

by subtracting $u$ from both sides of the identity in (24), dividing by $b$, and replacing $R$ by $R(t_g)$. Let $x_g$ or $t_g$ be specified so that $\frac{x_g - X_\nu}{X_q - X_p}$ can be calculated. Then a lower confidence bound on $R(t_g)$ is given by the value of $R$ corresponding to the $\underline{V}$ which is equal to $\frac{x_g - X_\nu}{X_q - X_p}$ and which represents a bound at the appropriate level on

$$V = \frac{\log \log (1/R) - Y_\nu}{Y_q - Y_p} = \frac{x_R - X_\nu}{X_q - X_p}$$ when $R$ is specified. It will often be the case, of course, that the required value of $\underline{V}$ is not available in the table, in which case a larger value can be used to provide a conservative bound on $R(t_g)$.

## EXTENSION OF RESULTS

### EXTENSION TO OTHER TWO-PARAMETER DISTRIBUTIONS

A natural extension of the results given above is the application of the methods used in obtaining confidence bounds to various other distributions. The fact that bounds based on three order statistics can be found for a percentage point $x_R$ of the distribution of $X$ (the extreme-value distribution of smallest values) depends, as noted above, upon the identity

$$X_\nu + \frac{y_R - Y_\nu}{Y_q - Y_p}(X_q - X_p) \equiv x_R = u + b \log\log(1/R), \quad 1 \le p < q \le m \le n \qquad (25)$$
$$1 < \nu < m < n,$$

where $Y_k = X_k - u)/b$ and $y_R = \log\log(1/R)$, with $R$ a specified proportion of the population to the right of the point $x_R$. For the extreme-value distribution, $R$ may represent a proportion of the population surviving at time $x_R$. The value of $y_R = -\log\log(1/R)$ has been tabulated in [15] for the extreme value distribution of <u>largest</u> values , but can also be calculated relatively easily. For other distributions, values of $y_R$ can often also be calculated or have been tabulated. For any distribution which can be transformed to one having a location-scale parameter, that is, with density of the form $f_{\theta_1,\theta_2}(x) = \frac{1}{\theta_2} g\left(\frac{x-\theta_1}{\theta_2}\right)$, for some $g$, the identity (25) holds. Three-

order-statistic bounds can therefore be obtained on $x_R$ for such a distribution if values of $y_R$ are obtainable and if percentiles of $(y_R-Y_v)/(Y_q-Y_p)$ can be calculated. If the first and second moments of the distribution of the order statistics of the distribution are available as well, a most efficient one of these bounds can be found for various combinations of $n$, $m$, $R$ and confidence level $1 - \alpha$. This method of obtaining bounds may be useful when samples from distributions for which classical bounds are available have been censored so that classical theory does not apply.

## EXTENSION TO THREE-PARAMETER DISTRIBUTIONS

A further extension of the above results is to a random variable $T - \lambda$, with X equal to the natural logarithm of $T - \lambda$, and the distribution of X dependent upon a location-scale parameter $\theta = (\theta_1, \theta_2)$. These include the three-parameter log normal, the three-parameter Pareto and the three-parameter Weibull distributions. The latter distribution is applicable to the situation in which a threshold $\lambda$ exists such that no failures can occur prior to the time $\lambda$. There is a proof given in Kaufman and Lipow [16] concerning confidence bounds on $t_R = \exp(x_R)$, which they have obtained as a function of order statistics $T_1$ and $T_n$ from any distribution of the type described above. The following is a generalization of that proof.

Consider a specified proportion $R = 1 - F(y_R)$

$= 1 - F\left(\frac{\log(t_R - \lambda) - \theta_1}{\theta_2}\right)$ of a population and a lower confidence bound $\underline{t_R}$ which can be obtained on a percentage point $t_R$ corresponding to $1 - R$ when $\lambda$ is zero. Let $\underline{t_R}$ be of the form $\exp(\underline{x_R}) = T_\nu(T_q/T_p)^{\underline{V}}$, $1 \le p < q \le m \le n$, $1 \le \nu \le m < n$. Then $\underline{t_R(\lambda)} = \lambda + (T_\nu - \lambda)\left[(T_q - \lambda)/(T_p - \lambda)\right]^{\underline{V}}$ gives a suitable bound for $t_R(\lambda) = \lambda + \exp(\theta_1+\theta_2 y_R)$ when $\lambda$ is known. It will be shown that $\underline{t_R}$ gives a conservative lower bound for $t_R(\lambda)$ as long as certain conditions are satisfied.

THEOREM I

If

$$P\left\{t_R(\lambda) = \lambda + \exp(\theta_1+\theta_2 y_R) \ge \underline{t_R(\lambda)} = \lambda + (T_\nu-\lambda)\left[(T_q-\lambda)/(T_p-\lambda)\right]^{\underline{V}}\right\} = 1-\alpha,$$

then $P\left\{t_R(\lambda) \ge \underline{t_R} = \exp(\underline{x_R})\right\} \ge 1-\alpha$, for $\underline{V}$ satisfying certain requirements dependent upon the relationship of $T_\nu$ to $T_p$ and $T_q$.

It is sufficient to demonstrate that $\underline{t_R} \le \underline{t_R(\lambda)}$ or that $\underline{t_R(\lambda)}$ is non-decreasing in $\lambda$. Consider

$$\underline{t_R(\lambda)} = \lambda + (T_\nu-\lambda)\left[(T_q-\lambda)/(T_p-\lambda)\right]^{\underline{V}}$$

The derivative of this expression with respect to $\lambda$ is given by

$$\frac{d\ t_R(\lambda)}{d\ \lambda} = 1 - \underline{V}\left[(T_q-\lambda)/(T_p-\lambda)^{\underline{V}-1}\right]\left[(T_\nu-\lambda)/T_p-\lambda)\right]$$

$$- \left[(T_q-\lambda)/(T_p-\lambda)\right]^{\underline{V}} + \underline{V}\left[(T_\nu-\lambda)/(T_p-\lambda)\right]\left[(T_q-\lambda)/(T_p-\lambda)\right]^{\underline{V}} \qquad (26)$$

$$= 1 - \left[(T_q-\lambda)/(T_p-\lambda)\right]^{\underline{V}} - \underline{V}\left[(T_\nu-\lambda)/(T_p-\lambda)\right]\left\{\left[(T_q-\lambda)/(T_p-\lambda)\right]^{\underline{V}-1}\right.$$

$$\left. - \left[(T_q-\lambda)/(T_p-\lambda)\right]^{\underline{V}}\right\}$$

Now let $\eta = (T_q-\lambda)/(T_p-\lambda)$, which is greater than or equal to 1 since $\lambda \le T_1 \le T_p \le T_q$. Then (26) can be written

$$1 - \eta^{\underline{V}} - \underline{V}\left[(T_\nu-\lambda)/(T_p-\lambda)\right]\eta^{\underline{V}-1}(1-\eta)$$

and it is required to demonstrate that this expression is non-negative or that

$$1 \ge \eta^{\underline{V}} + \underline{V}\left[(T_\nu-\lambda)/(T_p-\lambda)\right]\eta^{\underline{V}-1}(1-\eta). \qquad (27)$$

Assume $T_p \le T_\nu \le T_q$. Then if $\underline{V} \le 0$, $\underline{V}\ \eta^{\underline{V}-1}(1-\eta) \ge 0$ and the right side of (27) has its highest (or worst) value when $T_\nu$ is equal to $T_q$. $T_q$ is therefore substituted for $T_\nu$ in (27), which becomes

$$1 \ge \eta^{\underline{V}} + \underline{V}\ \eta^{\underline{V}}(1-\eta) \qquad (28)$$

Then for $\underline{V} \le 0$ the right side of (28) is equal to 1 and 0, respectively, for $\eta$ equal to 1 and $\infty$, respectively. Furthermore the right side of (28) is monotonically non-increasing in $\eta$ for $\eta \ge 1$,

if and only if $\underline{V} \leq -1$ for $\underline{V} \leq 0$. Thus, the inequality (26) is satisfied, $t_R(\lambda)$ is non-decreasing in $\lambda$, and $t_R$ is a conservative lower bound for $t_R$, for $T_p \leq T_\nu \leq T_q$ as long as a negative $\underline{V}$ is less than or equal to -1. If $\underline{V}$ is positive, then $T_p$ gives the worst value for $T_\nu$ and $1 + \Pi^{V} + \underline{V}\,\Pi^{V-1}(1-\Pi)$ is equal to 1 for $\Pi = 1$ and $-\infty$ for $\Pi = \infty$ and is monotonically non-increasing in $\Pi$ for $\Pi \geq 1$ if and only if $\underline{V} \geq 1$. For $T_\nu \leq T_p$ and for $T_\nu \geq T_q$ it can be shown by the method used above that $t_R$ is a conservative lower bound for $t_R$ if $\underline{V} \leq 0$ and if $\underline{V} \geq 0$, respectively.

## CALCULATION OF THE COEFFICIENTS YIELDING THE EXACT BOUNDS AND USE OF THE TABLES

To obtain the coefficient of an exact upper confidence bound for b at level $1 - \alpha$ based on $X_q$ and $X_p$ one may proceed from (3) as in obtaining the power function. The coefficient $LC_{p,q}(\alpha)$ defining the bound is thus given by iteratively solving the equation

$$1 - \frac{n!}{(p-1)!(q-p-1)!(n-q)!}\sum_{k=0}^{p-1}\sum_{j=0}^{q-p-1}(-1)^{k+1}\binom{p-1}{k}\binom{q-p-1}{j}\frac{1/(n-q+j+1)}{((q-p-j+k)+(n-q+j+1)\;C_{p,q}(\alpha))} = \alpha, \tag{29}$$

where $C_{p,q}(\alpha) = \exp\,(LC_{p,q}(\alpha))$.

The coefficient defining an exact upper bound for $x_R$ at confidence level $1 - \alpha^*$, as described above, is the $100\alpha^*$ percent point of the

distribution of $V = \dfrac{\log\log(1/R) - X_\nu}{X_q - X_p}$. The expression from which the distribution percentiles of $V$ may be obtained differs accordingly as $\nu < p$, $\nu = p$, $p < \nu < q$, $\nu = q$, or $\nu > q$. For example, if $p < \nu < q$, the solution to

$$\int_1^\infty \frac{n!}{(p-1)!(\nu-p-1)\nu(q-\nu-1)!(n-q)!} \sum_{k=o}^{p-1} \sum_{j=o}^{\nu-p-1} \sum_{m=o}^{q-\nu-1} (-1)^{j+k+m} \binom{p-1}{k}\binom{\nu-p-1}{j}\binom{q-\nu-1}{m}$$

$$\Big\{ -cs \exp\{-c\,(n-p+k+1)s\}/(n-q+m+1)(n-\nu+m-j+k+1)s$$

$$+ c\,s \exp\left\{-c\left[(q-p-m+k)\,s + (n-q+m+1)\,s^{1-1/v}\right]\right\}/(n-q+m+1)\left[(q-\nu-j+k)\,s + (n-q+m+1)\,s^{1-1/v}\right]$$

$$+ c\,s \exp\left\{-c\left[(n+\nu-q-p+j+1)s+(q-\nu-j+k)s^{1+1/v}\right]\right\}/(q-\nu-j+k)\left[(q-\nu-j+k)s^{1+1/v} + (n-q+m+1)s\right]$$

$$- c\,s \exp\{c\,(n+\nu-p+k+1)s\}/(q-\nu-j+k)(n-\nu+m-j+k+1)s\Big\}\, d\,s = \alpha,$$

where $c = \log(1/R)$, defines the appropriate $v = \underline{V}$ when $\underline{V}$ is negative or zero. For $0 < \underline{V} < \infty$, the limits of integration for $s$ are $0$ to $1$, and $1 - \alpha$ rather than $\alpha$ appears on the right side of the equation. The equations to be solved for $\underline{V}$ under other restrictions on $\nu$ are slightly less complicated.

The values of $LC_{p,q}(\alpha) = LC_1(\alpha)$ for $X_p = X_1$ were calculated iteratively by means of both equation (29) and equation (9) so that any loss of accuracy with increasing number of terms of (29) could be determined. The calculations were made in Fortran IV built-in double precision (16 significant figures), and the values of $\exp\left[LC_1(\alpha)\right]$ calculated agreed to at least seven significant figures no matter how large the number of terms involved, through $n$ equal to 20. An additional external check is given by the values of percentiles of

$$\frac{T_p^{1/b}}{T_q^{1/b} - T_p^{1/b}} = \left[\left(T_q/T_p\right)^{1/b} - 1\right]^{-1} \quad \text{for } 2 \le n \le 10,\ 1 \le p < q \le n,$$

exhibited in Table I of [17]. The values of $LC$ given in Table B.I of this report are thus accurate to within a unit in the final decimal place shown.

Characteristic of the values of ratios of mean squared errors of best linear invariant estimators of $b$ (based on all $m$ order statistics and on the combination of $X_p$ and $X_q$ yielding a most efficient bound) is 0.793 for $n = m = 17$. For the same values of $n$ and $m$, the corresponding ratio associated with the $\nu$, $p$, and $q$ yielding a best invariant estimator of $x_{.90}$ of the form $X_\nu + C\,(X_q - X_p)$ is 0.877. The additional information contained in the $\nu^{th}$ order statistic appears to contribute considerably to the efficiency of estimators and bounds.

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**TABLE B.I** - VALUES OF P, Q, AND LC FOR OBTAINING MOST EFFICIENT EXACT CONFIDENCE BOUNDS OF THE FORM $(\log T_Q - \log T_P) / LC$, FOR THE WEIBULL SHAPE PARAMETER, BASED ON ANY TWO OF THE FIRST M OF N WEIBULL ORDER STATISTICS, $T_1 \leq T_2 \leq \ldots \leq T_M \leq \ldots \leq T_N$

| | | CONFIDENCE LEVEL | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.99 | | | 0.95 | | | 0.90 | | |
| N | M | P | Q | LC | P | Q | LC | P | Q | LC |
| 2 | 2 | 1 | 2 | 0.020 | 1 | 2 | 0.100 | 1 | 2 | 0.201 |
| 3 | 2 | 1 | 2 | 0.015 | 1 | 2 | 0.076 | 1 | 2 | 0.154 |
| | 3 | 1 | 3 | 0.213 | 1 | 3 | 0.482 | 1 | 3 | 0.693 |
| 4 | 2 | 1 | 2 | 0.013 | 1 | 2 | 0.068 | 1 | 2 | 0.138 |
| | 3 | 1 | 3 | 0.167 | 1 | 3 | 0.387 | 1 | 3 | 0.565 |
| | 4 | 1 | 4 | 0.481 | 1 | 4 | 0.847 | 1 | 4 | 1.099 |
| 5 | 2 | 1 | 2 | 0.013 | 1 | 2 | 0.064 | 1 | 2 | 0.130 |
| | 3 | 1 | 3 | 0.149 | 1 | 3 | 0.348 | 1 | 3 | 0.513 |
| | 4 | 1 | 4 | 0.392 | 1 | 4 | 0.706 | 1 | 4 | 0.928 |
| | 5 | 1 | 5 | 0.737 | 1 | 5 | 1.151 | 1 | 5 | 1.419 |
| 6 | 2 | 1 | 2 | 0.012 | 1 | 2 | 0.061 | 1 | 2 | 0.125 |
| | 3 | 1 | 3 | 0.139 | 1 | 3 | 0.327 | 1 | 3 | 0.483 |
| | 4 | 1 | 4 | 0.353 | 1 | 4 | 0.642 | 1 | 4 | 0.851 |
| | 5 | 1 | 5 | 0.617 | 1 | 5 | 0.983 | 1 | 5 | 1.226 |
| | 6 | 1 | 6 | 0.964 | 1 | 6 | 1.403 | 1 | 6 | 1.680 |
| 7 | 2 | 1 | 2 | 0.012 | 1 | 2 | 0.060 | 1 | 2 | 0.122 |
| | 3 | 1 | 3 | 0.133 | 1 | 3 | 0.314 | 1 | 3 | 0.465 |
| | 4 | 1 | 4 | 0.330 | 1 | 4 | 0.605 | 1 | 4 | 0.805 |
| | 5 | 1 | 5 | 0.560 | 1 | 5 | 0.902 | 1 | 5 | 1.133 |
| | 6 | 1 | 6 | 0.824 | 1 | 6 | 1.219 | 1 | 6 | 1.474 |
| | 7 | 1 | 7 | 1.164 | 1 | 7 | 1.618 | 2 | 7 | 1.284 |
| 8 | 2 | 1 | 2 | 0.011 | 1 | 2 | 0.058 | 1 | 2 | 0.120 |
| | 3 | 1 | 3 | 0.129 | 1 | 3 | 0.304 | 1 | 3 | 0.452 |
| | 4 | 1 | 4 | 0.315 | 1 | 4 | 0.581 | 1 | 4 | 0.775 |
| | 5 | 1 | 5 | 0.526 | 1 | 5 | 0.853 | 1 | 5 | 1.076 |
| | 6 | 1 | 6 | 0.754 | 1 | 6 | 1.127 | 1 | 6 | 1.371 |
| | 7 | 1 | 7 | 1.010 | 1 | 7 | 1.424 | 2 | 7 | 1.101 |
| | 8 | 1 | 8 | 1.340 | 2 | 8 | 1.263 | 2 | 8 | 1.479 |
| 9 | 2 | 1 | 2 | 0.011 | 1 | 2 | 0.056 | 1 | 2 | 0.118 |
| | 3 | 1 | 3 | 0.125 | 1 | 3 | 0.298 | 1 | 3 | 0.443 |
| | 4 | 1 | 4 | 0.305 | 1 | 4 | 0.564 | 1 | 4 | 0.754 |
| | 5 | 1 | 5 | 0.503 | 1 | 5 | 0.820 | 1 | 5 | 1.037 |
| | 6 | 1 | 6 | 0.710 | 1 | 6 | 1.069 | 1 | 6 | 1.306 |
| | 7 | 1 | 7 | 0.930 | 1 | 7 | 1.323 | 2 | 7 | 1.008 |
| | 8 | 1 | 8 | 1.177 | 2 | 8 | 1.091 | 2 | 8 | 1.286 |
| | 9 | 2 | 9 | 1.067 | 2 | 9 | 1.430 | 2 | 9 | 1.649 |
| 10 | 2 | 1 | 2 | 0.011 | 1 | 2 | 0.057 | 1 | 2 | 0.116 |
| | 3 | 1 | 3 | 0.123 | 1 | 3 | 0.292 | 1 | 3 | 0.435 |
| | 4 | 1 | 4 | 0.297 | 1 | 4 | 0.551 | 1 | 4 | 0.738 |
| | 5 | 1 | 5 | 0.486 | 1 | 5 | 0.796 | 1 | 5 | 1.009 |
| | 6 | 1 | 6 | 0.680 | 1 | 6 | 1.029 | 1 | 6 | 1.261 |
| | 7 | 1 | 7 | 0.876 | 1 | 7 | 1.259 | 2 | 7 | 0.950 |
| | 8 | 1 | 8 | 1.089 | 2 | 8 | 1.002 | 2 | 8 | 1.186 |
| | 9 | 2 | 9 | 0.923 | 2 | 9 | 1.250 | 2 | 9 | 1.450 |
| | 10 | [illegible] | 10 | 1.209 | 2 | 10 | 1.579 | 2 | 10 | 1.801 |
| 11 | 2 | 1 | 2 | 0.011 | 1 | 2 | 0.056 | 1 | 2 | 0.115 |
| | 3 | 1 | 3 | 0.121 | 1 | 3 | 0.288 | 1 | 3 | 0.429 |
| | 4 | 1 | 4 | 0.291 | 1 | 4 | 0.541 | 1 | 4 | 0.725 |
| | 5 | 1 | 5 | 0.473 | 1 | 5 | 0.778 | 1 | 5 | 0.988 |
| | 6 | 1 | 6 | 0.658 | 1 | 6 | 1.000 | 1 | 6 | 1.228 |
| | 7 | 1 | 7 | 0.843 | 1 | 7 | 1.214 | 2 | 7 | 0.910 |
| | 8 | 1 | 8 | 1.032 | 2 | 8 | 0.944 | 2 | 8 | 1.122 |
| | 9 | 2 | 9 | 0.846 | 2 | 9 | 1.154 | 2 | 9 | 1.349 |
| | 10 | 2 | 10 | 1.057 | 2 | 10 | 1.394 | 2 | 10 | 1.598 |
| | 11 | 2 | 11 | 1.338 | 2 | 11 | 1.713 | 2 | 11 | 1.936 |
| 12 | 2 | 1 | 2 | 0.011 | 1 | 2 | 0.056 | 1 | 2 | 0.114 |
| | 3 | 1 | 3 | 0.120 | 1 | 3 | 0.285 | 1 | 3 | 0.425 |
| | 4 | 1 | 4 | 0.286 | 1 | 4 | 0.532 | 1 | 4 | 0.715 |
| | 5 | 1 | 5 | 0.463 | 1 | 5 | 0.763 | 1 | 5 | 0.971 |
| | 6 | 1 | 6 | 0.641 | 1 | 6 | 0.977 | 1 | 6 | 1.202 |
| | 7 | 1 | 7 | 0.816 | 1 | 7 | 1.180 | 2 | 7 | 0.880 |
| | 8 | 1 | 8 | 0.992 | 2 | 8 | 0.904 | 2 | 8 | 1.077 |
| | 9 | 2 | 9 | 0.796 | 2 | 9 | 1.092 | 2 | 9 | 1.276 |
| | 10 | 2 | 10 | 0.974 | 2 | 10 | 1.293 | 2 | 10 | 1.488 |
| | 11 | 2 | 11 | 1.181 | 2 | 11 | 1.524 | 2 | 11 | 1.730 |
| | 12 | 2 | 12 | 1.457 | 2 | 12 | 1.835 | 2 | 12 | 2.059 |

## TABLE B.I - CONTINUED

CONFIDENCE LEVEL

| N | M | 0.99 P | 0.99 Q | 0.99 LC | 0.95 P | 0.95 Q | 0.95 LC | 0.90 P | 0.90 Q | 0.90 LC |
|---|---|---|---|---|---|---|---|---|---|---|
| 13 | 2 | 1 | 2 | 0.011 | 1 | 2 | 0.055 | 1 | 2 | 0.114 |
| | 3 | 1 | 3 | 0.118 | 1 | 3 | 0.282 | 1 | 3 | 0.421 |
| | 4 | 1 | 4 | 0.282 | 1 | 4 | 0.526 | 1 | 4 | 0.707 |
| | 5 | 1 | 5 | 0.455 | 1 | 5 | 0.751 | 1 | 5 | 0.957 |
| | 6 | 1 | 6 | 0.627 | 1 | 6 | 0.959 | 1 | 6 | 1.182 |
| | 7 | 1 | 7 | 0.795 | 1 | 7 | 1.154 | 2 | 7 | 0.857 |
| | 8 | 1 | 8 | 0.961 | 2 | 8 | 0.874 | 2 | 8 | 1.043 |
| | 9 | 2 | 9 | 0.760 | 2 | 9 | 1.047 | 2 | 9 | 1.227 |
| | 10 | 2 | 10 | 0.919 | 2 | 10 | 1.227 | 2 | 10 | 1.416 |
| | 11 | 2 | 11 | 1.093 | 2 | 11 | 1.420 | 2 | 11 | 1.618 |
| | 12 | 2 | 12 | 1.295 | 2 | 12 | 1.643 | 3 | 12 | 1.491 |
| | 13 | 2 | 13 | 1.566 | 2 | 13 | 1.947 | 3 | 13 | 1.801 |
| 14 | 2 | 1 | 2 | 0.011 | 1 | 2 | 0.055 | 1 | 2 | 0.113 |
| | 3 | 1 | 3 | 0.117 | 1 | 3 | 0.280 | 1 | 3 | 0.418 |
| | 4 | 1 | 4 | 0.279 | 1 | 4 | 0.520 | 1 | 4 | 0.700 |
| | 5 | 1 | 5 | 0.449 | 1 | 5 | 0.742 | 1 | 5 | 0.946 |
| | 6 | 1 | 6 | 0.616 | 1 | 6 | 0.944 | 1 | 6 | 1.165 |
| | 7 | 1 | 7 | 0.778 | 1 | 7 | 1.133 | 2 | 7 | 0.839 |
| | 8 | 1 | 8 | 0.937 | 2 | 8 | 0.850 | 2 | 8 | 1.017 |
| | 9 | 2 | 9 | 0.733 | 2 | 9 | 1.014 | 2 | 9 | 1.190 |
| | 10 | 2 | 10 | 0.880 | 2 | 10 | 1.179 | 2 | 10 | 1.363 |
| | 11 | 2 | 11 | 1.034 | 2 | 11 | 1.350 | 2 | 11 | 1.543 |
| | 12 | 2 | 12 | 1.203 | 2 | 12 | 1.536 | 3 | 12 | 1.384 |
| | 13 | 2 | 13 | 1.401 | 3 | 13 | 1.427 | 3 | 13 | 1.605 |
| | 14 | 2 | 14 | 1.667 | 3 | 14 | 1.713 | 3 | 14 | 1.907 |
| 15 | 2 | 1 | 2 | 0.011 | 1 | 2 | 0.055 | 1 | 2 | 0.112 |
| | 3 | 1 | 3 | 0.116 | 1 | 3 | 0.278 | 1 | 3 | 0.415 |
| | 4 | 1 | 4 | 0.276 | 1 | 4 | 0.516 | 1 | 4 | 0.694 |
| | 5 | 1 | 5 | 0.443 | 1 | 5 | 0.734 | 1 | 5 | 0.937 |
| | 6 | 1 | 6 | 0.607 | 1 | 6 | 0.932 | 1 | 6 | 1.151 |
| | 7 | 1 | 7 | 0.765 | 1 | 7 | 1.116 | 2 | 7 | 0.824 |
| | 8 | 1 | 8 | 0.918 | 2 | 8 | 0.831 | 2 | 8 | 0.995 |
| | 9 | 2 | 9 | 0.712 | 2 | 9 | 0.987 | 2 | 9 | 1.161 |
| | 10 | 2 | 10 | 0.849 | 2 | 10 | 1.142 | 2 | 10 | 1.323 |
| | 11 | 2 | 11 | 0.991 | 2 | 11 | 1.300 | 2 | 11 | 1.488 |
| | 12 | 2 | 12 | 1.142 | 2 | 12 | 1.464 | 3 | 12 | 1.313 |
| | 13 | 2 | 13 | 1.306 | 3 | 13 | 1.326 | 3 | 13 | 1.495 |
| | 14 | 2 | 14 | 1.499 | 3 | 14 | 1.530 | 3 | 14 | 1.709 |
| | 15 | 3 | 15 | 1.476 | 3 | 15 | 1.811 | 3 | 15 | 2.005 |
| 16 | 2 | 1 | 2 | 0.011 | 1 | 2 | 0.055 | 1 | 2 | 0.112 |
| | 3 | 1 | 3 | 0.116 | 1 | 3 | 0.276 | 1 | 3 | 0.413 |
| | 4 | 1 | 4 | 0.273 | 1 | 4 | 0.512 | 1 | 4 | 0.689 |
| | 5 | 1 | 5 | 0.439 | 1 | 5 | 0.727 | 1 | 5 | 0.929 |
| | 6 | 1 | 6 | 0.599 | 1 | 6 | 0.922 | 1 | 6 | 1.139 |
| | 7 | 1 | 7 | 0.753 | 1 | 7 | 1.101 | 2 | 7 | 0.812 |
| | 8 | 1 | 8 | 0.902 | 2 | 8 | 0.816 | 2 | 8 | 0.978 |
| | 9 | 2 | 9 | 0.695 | 2 | 9 | 0.966 | 2 | 9 | 1.137 |
| | 10 | 2 | 10 | 0.825 | 2 | 10 | 1.113 | 2 | 10 | 1.292 |
| | 11 | 2 | 11 | 0.959 | 2 | 11 | 1.260 | 2 | 11 | 1.446 |
| | 12 | 2 | 12 | 1.096 | 2 | 12 | 1.411 | 3 | 12 | 1.261 |
| | 13 | 2 | 13 | 1.242 | 3 | 13 | 1.258 | 3 | 13 | 1.421 |
| | 14 | 2 | 14 | 1.403 | 3 | 14 | 1.427 | 3 | 14 | 1.598 |
| | 15 | 3 | 15 | 1.319 | 3 | 15 | 1.626 | 3 | 15 | 1.806 |
| | 16 | 3 | 16 | 1.565 | 3 | 16 | 1.901 | 3 | 16 | 2.097 |
| 17 | 2 | 1 | 2 | 0.011 | 1 | 2 | 0.054 | 1 | 2 | 0.112 |
| | 3 | 1 | 3 | 0.115 | 1 | 3 | 0.275 | 1 | 3 | 0.411 |
| | 4 | 1 | 4 | 0.271 | 1 | 4 | 0.508 | 1 | 4 | 0.685 |
| | 5 | 1 | 5 | 0.435 | 1 | 5 | 0.721 | 1 | 5 | 0.922 |
| | 6 | 1 | 6 | 0.593 | 1 | 6 | 0.913 | 1 | 6 | 1.129 |
| | 7 | 1 | 7 | 0.744 | 1 | 7 | 1.089 | 2 | 7 | 0.801 |
| | 8 | 1 | 8 | 0.888 | 2 | 8 | 0.803 | 2 | 8 | 0.963 |
| | 9 | 2 | 9 | 0.680 | 2 | 9 | 0.948 | 2 | 9 | 1.117 |
| | 10 | 2 | 10 | 0.806 | 2 | 10 | 1.089 | 2 | 10 | 1.266 |
| | 11 | 2 | 11 | 0.932 | 2 | 11 | 1.229 | 2 | 11 | 1.412 |
| | 12 | 2 | 12 | 1.061 | 2 | 12 | 1.370 | 3 | 12 | 1.221 |
| | 13 | 2 | 13 | 1.194 | 3 | 13 | 1.207 | 3 | 13 | 1.367 |
| | 14 | 2 | 14 | 1.336 | 3 | 14 | 1.356 | 3 | 14 | 1.522 |
| | 15 | 3 | 15 | 1.228 | 3 | 15 | 1.521 | 3 | 15 | 1.693 |
| | 16 | 3 | 16 | 1.406 | 3 | 16 | 1.716 | 3 | 16 | 1.897 |
| | 17 | 3 | 17 | 1.649 | 3 | 17 | 1.986 | 3 | 17 | 2.182 |

## TABLE B.I - CONTINUED

| | | CONFIDENCE LEVEL 0.99 | | | CONFIDENCE LEVEL 0.95 | | | CONFIDENCE LEVEL 0.90 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| N | M | P | Q | LC | P | Q | LC | P | Q | LC |
| 18 | 2 | 1 | 2 | 0.011 | 1 | 2 | 0.054 | 1 | 2 | 0.111 |
| | 3 | 1 | 3 | 0.114 | 1 | 3 | 0.273 | 1 | 3 | 0.409 |
| | 4 | 1 | 4 | 0.270 | 1 | 4 | 0.505 | 1 | 4 | 0.681 |
| | 5 | 1 | 5 | 0.431 | 1 | 5 | 0.716 | 1 | 5 | 0.916 |
| | 6 | 1 | 6 | 0.587 | 1 | 6 | 0.905 | 1 | 6 | 1.121 |
| | 7 | 1 | 7 | 0.735 | 1 | 7 | 1.078 | 2 | 7 | 0.792 |
| | 8 | 1 | 8 | 0.877 | 2 | 8 | 0.792 | 2 | 8 | 0.951 |
| | 9 | 2 | 9 | 0.669 | 2 | 9 | 0.933 | 2 | 9 | 1.101 |
| | 10 | 2 | 10 | 0.790 | 2 | 10 | 1.070 | 2 | 10 | 1.244 |
| | 11 | 2 | 11 | 0.911 | 2 | 11 | 1.204 | 2 | 11 | 1.384 |
| | 12 | 2 | 12 | 1.033 | 2 | 12 | 1.337 | 3 | 12 | 1.189 |
| | 13 | 2 | 13 | 1.157 | 3 | 13 | 1.168 | 3 | 13 | 1.325 |
| | 14 | 2 | 14 | 1.286 | 3 | 14 | 1.304 | 3 | 14 | 1.466 |
| | 15 | 3 | 15 | 1.165 | 3 | 15 | 1.448 | 3 | 15 | 1.616 |
| | 16 | 3 | 16 | 1.313 | 3 | 16 | 1.609 | 3 | 16 | 1.783 |
| | 17 | 3 | 17 | 1.488 | 3 | 17 | 1.800 | 3 | 17 | 1.981 |
| | 18 | 3 | 28 | 1.727 | 3 | 18 | 2.066 | 3 | 18 | 2.261 |
| 19 | 2 | 1 | 2 | 0.011 | 1 | 2 | 0.054 | 1 | 2 | 0.111 |
| | 3 | 1 | 3 | 0.114 | 1 | 3 | 0.272 | 1 | 3 | 0.407 |
| | 4 | 1 | 4 | 0.268 | 1 | 4 | 0.503 | 1 | 4 | 0.678 |
| | 5 | 1 | 5 | 0.428 | 1 | 5 | 0.711 | 1 | 5 | 0.910 |
| | 6 | 1 | 6 | 0.582 | 1 | 6 | 0.899 | 1 | 6 | 1.113 |
| | 7 | 1 | 7 | 0.728 | 1 | 7 | 1.069 | 2 | 7 | 0.784 |
| | 8 | 1 | 8 | 0.867 | 2 | 8 | 0.782 | 2 | 8 | 0.940 |
| | 9 | 2 | 9 | 0.658 | 2 | 9 | 0.920 | 2 | 9 | 1.086 |
| | 10 | 2 | 10 | 0.776 | 2 | 10 | 1.053 | 2 | 10 | 1.226 |
| | 11 | 2 | 11 | 0.893 | 2 | 11 | 1.182 | 2 | 11 | 1.361 |
| | 12 | 2 | 12 | 1.009 | 2 | 12 | 1.310 | 3 | 12 | 1.162 |
| | 13 | 2 | 13 | 1.127 | 3 | 13 | 1.137 | 3 | 13 | 1.291 |
| | 14 | 2 | 14 | 1.247 | 3 | 14 | 1.263 | 3 | 14 | 1.423 |
| | 15 | 3 | 15 | 1.118 | 3 | 15 | 1.394 | 3 | 15 | 1.559 |
| | 16 | 3 | 16 | 1.248 | 3 | 16 | 1.535 | 3 | 16 | 1.704 |
| | 17 | 3 | 17 | 1.393 | 3 | 17 | 1.691 | 4 | 17 | 1.608 |
| | 18 | 3 | 18 | 1.565 | 3 | 18 | 1.878 | 4 | 18 | 1.799 |
| | 19 | 3 | 19 | 1.802 | 3 | 19 | 2.141 | 4 | 19 | 2.069 |
| 20 | 2 | 1 | 2 | 0.011 | 1 | 2 | 0.054 | 1 | 2 | 0.111 |
| | 3 | 1 | 3 | 0.113 | 1 | [illegible] | 0.271 | 1 | 3 | 0.406 |
| | 4 | 1 | 4 | 0.267 | 1 | 4 | 0.500 | 1 | 4 | 0.675 |
| | 5 | 1 | 5 | 0.425 | 1 | 5 | 0.707 | 1 | 5 | 0.905 |
| | 6 | 1 | 6 | 0.578 | 1 | 6 | 0.893 | 1 | 6 | 1.106 |
| | 7 | 1 | 7 | 0.722 | 1 | 7 | 1.061 | 2 | 7 | 0.777 |
| | 8 | 1 | 8 | 0.858 | 2 | 8 | 0.774 | 2 | 8 | 0.931 |
| | 9 | 2 | 9 | 0.650 | 2 | 9 | 0.909 | 2 | 9 | 1.074 |
| | 10 | 2 | 10 | 0.765 | 2 | 10 | 1.039 | 2 | 10 | 1.211 |
| | 11 | 2 | 11 | 0.878 | 2 | 11 | 1.164 | 2 | 11 | 1.342 |
| | 12 | 2 | 12 | 0.990 | 2 | 12 | 1.287 | 3 | 12 | 1.140 |
| | 13 | 2 | 13 | 1.102 | 3 | 13 | 1.111 | 3 | 13 | 1.264 |
| | 14 | 2 | 14 | 1.216 | 3 | 14 | 1.230 | 3 | 14 | 1.387 |
| | 15 | 3 | 15 | 1.081 | 3 | 15 | 1.352 | 3 | 15 | 1.514 |
| | 16 | 3 | 16 | 1.199 | 3 | 16 | 1.480 | 3 | 16 | 1.646 |
| | 17 | 3 | 17 | 1.327 | 3 | 17 | 1.616 | 4 | 17 | 1.532 |
| | 18 | 3 | 18 | 1.469 | 4 | 18 | 1.532 | 4 | 18 | 1.688 |
| | 19 | 3 | 19 | 1.638 | 4 | 19 | 1.712 | 4 | 19 | 1.875 |
| | 20 | 3 | 20 | 1.872 | 4 | 20 | 1.964 | 4 | 20 | 2.141 |

## TABLE B.I - CONTINUED

| | | CONFIDENCE LEVEL | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.99 | | | 0.95 | | | 0.90 | | |
| N | M | P | Q | LC | P | Q | LC | P | Q | LC |
| 21 | 2 | 1 | 2 | 0.011 | 1 | 2 | 0.054 | 1 | 2 | 0.110 |
| | 3 | 1 | 3 | 0.113 | 1 | 3 | 0.270 | 1 | 3 | 0.404 |
| | 4 | 1 | 4 | 0.265 | 1 | 4 | 0.498 | 1 | 4 | 0.672 |
| | 5 | 1 | 5 | 0.423 | 1 | 5 | 0.704 | 1 | 5 | 0.901 |
| | 6 | 1 | 6 | 0.574 | 1 | 6 | 0.887 | 1 | 6 | 1.100 |
| | 7 | 1 | 7 | 0.716 | 1 | 7 | 1.053 | 2 | 7 | 0.771 |
| | 8 | 1 | 8 | 0.851 | 2 | 8 | 0.767 | 2 | 8 | 0.922 |
| | 9 | 2 | 9 | 0.642 | 2 | 9 | 0.899 | 2 | 9 | 1.063 |
| | 10 | 2 | 10 | 0.754 | 2 | 10 | 1.025 | 2 | 10 | 1.197 |
| | 11 | 2 | 11 | 0.863 | 2 | 11 | 1.148 | 2 | 11 | 1.325 |
| | 12 | 2 | 12 | 0.973 | 2 | 12 | 1.267 | 3 | 12 | 1.122 |
| | 13 | 2 | 13 | 1.081 | 3 | 13 | 1.090 | 3 | 13 | 1.240 |
| | 14 | 2 | 14 | 1.190 | 3 | 14 | 1.203 | 3 | 14 | 1.358 |
| | 15 | 3 | 15 | 1.051 | 3 | 15 | 1.318 | 3 | 15 | 1.477 |
| | 16 | 3 | 16 | 1.161 | 3 | 16 | 1.436 | 3 | 16 | 1.599 |
| | 17 | 3 | 17 | 1.277 | 3 | 17 | 1.560 | 4 | 17 | 1.475 |
| | 18 | 3 | 18 | 1.401 | 4 | 18 | 1.459 | 4 | 18 | 1.611 |
| | 19 | 3 | 19 | 1.541 | 4 | 19 | 1.606 | 4 | 19 | 1.763 |
| | 20 | 3 | 20 | 1.708 | 4 | 20 | 1.783 | 4 | 20 | 1.947 |
| | 21 | 3 | 21 | 1.939 | 4 | 21 | 2.032 | 4 | 21 | 2.210 |
| 22 | 2 | 1 | 2 | 0.011 | 1 | 2 | 0.054 | 1 | 2 | 0.110 |
| | 3 | 1 | 3 | 0.113 | 1 | 3 | 0.269 | 1 | 3 | 0.463 |
| | 4 | 1 | 4 | 0.264 | 1 | 4 | 0.496 | 1 | 4 | 0.670 |
| | 5 | 1 | 5 | 0.421 | 1 | 5 | 0.700 | 1 | 5 | 0.897 |
| | 6 | 1 | 6 | 0.570 | 1 | 6 | 0.883 | 1 | 6 | 1.095 |
| | 7 | 1 | 7 | 0.711 | 1 | 7 | 1.047 | 2 | 7 | 0.766 |
| | 8 | 1 | 8 | 0.844 | 2 | 8 | 0.760 | 2 | 8 | 0.915 |
| | 9 | 2 | 9 | 0.635 | 2 | 9 | 0.891 | 2 | 9 | 1.054 |
| | 10 | 2 | 10 | 0.746 | 2 | 10 | 1.015 | 2 | 10 | 1.185 |
| | 11 | 2 | 11 | 0.853 | 2 | 11 | 1.135 | 2 | 11 | 1.310 |
| | 12 | 2 | 12 | 0.959 | 2 | 12 | 1.251 | 3 | 12 | 1.106 |
| | 13 | 2 | 13 | 1.063 | 3 | 13 | 1.071 | 3 | 13 | 1.220 |
| | 14 | 2 | 14 | 1.168 | 3 | 14 | 1.180 | 3 | 14 | 1.333 |
| | 15 | 3 | 15 | 1.027 | 3 | 15 | 1.289 | 3 | 15 | 1.447 |
| | 16 | 3 | 16 | 1.130 | 3 | 16 | 1.401 | 3 | 16 | 1.562 |
| | 17 | 3 | 17 | 1.237 | 3 | 17 | 1.515 | 4 | 17 | 1.430 |
| | 18 | 3 | 18 | 1.350 | 4 | 18 | 1.405 | 4 | 18 | 1.553 |
| | 19 | 3 | 19 | 1.472 | 4 | 19 | 1.533 | 4 | 19 | 1.655 |
| | 20 | 4 | 20 | 1.405 | 4 | 20 | 1.677 | 4 | 20 | 1.835 |
| | 21 | 4 | 21 | 1.565 | 4 | 21 | 1.851 | 4 | 21 | 2.016 |
| | 22 | 4 | 22 | 1.767 | 4 | 22 | 2.097 | 4 | 22 | 2.274 |

# APPENDIX C

## TABLES FOR OBTAINING THE BEST LINEAR INVARIANT ESTIMATES OF PARAMETERS OF THE WEIBULL DISTRIBUTION


### SUMMARY

A censored life-test situation is considered and the assumption of a Weibull distribution for failure times is made. Tables are given for estimating log reliable life, where the estimator is best among linear estimators with expected loss invariant under translations. These best linear invariant (BLI) estimators have uniformly smaller expected loss than the Gauss-Markov best linear unbiased (BLU) estimators and are simple linear functions of the BLU estimators. The preliminary discussion involves a comparison of the BLI estimators with other widely used estimators. Solutions to the problem of obtaining confidence bounds are also discussed.

# INTRODUCTION

Assume a random sample of $n$ items is subjected to life test until $m \leq n$ failures occur. Assume further that the failure times associated with the $n$ items are from a population of random variables identically distributed according to a two-parameter Weibull law. Let the random variable $T$ represent failure time in this population. Then

$$\Pr[T \leq t] \equiv F_{\delta,b}(t) = 1 - \exp\{-(t/\delta)^{1/b}\} \quad ,$$

for $t \geq 0$, and $F_{\delta,b}(t) = 0$ otherwise. Both parameters $\delta$ and $b$ are positive. The logarithms of the failure times have the extreme-value distribution of smallest values. If $X = \log T$ and $u = \log \delta$, then

$$\Pr[X \leq x] \equiv F_{u,b}(x) = 1 - \exp\{-\exp[(x-u)/b]\} \quad .$$

The parameter $b$ is the scale parameter of the distribution of $X$; the variance of $X$ is $\frac{\pi^2 b^2}{6}$. The parameter $u$ is a location parameter, the mode of the probability density function associated with the distribution of $X$.

The reliability function, which gives the proportion of the population surviving at log time $x$, is given by $R(x) = 1-F(x)$, or

$$R_{u,b}(x) = \exp\{-\exp[(x-u)/b]\} \quad . \tag{1}$$

Suppose that a value $R$ is specified for $R_{u,b}(x)$ and that the problem of interest is the estimation of the value of $x$, call it $x_R$, corresponding to the specified $R$ on the basis of a censored sample. The value $x_R$ is simply the log time at which 100R% of the population will have survived, and is often referred to as log reliable life. Since, from (1), $R = \exp\{-\exp[(x_R-u)/b]\}$, or $x_R = u + b \log\log(1/R)$, $x_R$ is a parametric function of the location parameter $u$ and the scale parameter $b$. In this report methods for estimating the general parametric function $\varphi = \ell_1 u + \ell_2 b$ (which includes $u$, $b$, and $x_R$) are considered.

## MAXIMUM-LIKELIHOOD ESTIMATION

Because of the regularity properties of the Weibull density function (see Kimball [1]), maximum-likelihood estimators of Weibull parameters enjoy the properties of consistency, asymptotic efficiency, asymptotic unbiasedness and asymptotic normality. The maximum-likelihood estimates cannot, however, be calculated explicitly, but must be determined by iterative procedures applied to sample data. Since this is so, the expected squared error of a Weibull maximum-likelihood estimator cannot be determined except for large samples, for which the variances of these asymptotically unbiased estimators are given by the Cramér-Rao bounds for regular unbiased estimators. Thus, for small sample sizes, the expected loss of the estimator cannot be compared with that of other estimators whose exact mean squared error can be calculated. Furthermore, in [2] it is shown that when no censoring occurs, the $n$ order statistics, $\{\log T_{(1,n)} \le \log T_{(2,n)} \le \cdots \le \log T_{(n,n)}\} = \{X_{(1,n)} \le X_{(2,n)} \le \cdots \le X_{(n,n)}\}$ from a size $n$ sample are minimally sufficient for $(u,b)$. They are, therefore, minimally sufficient for $(\delta,b)$ or for any couple consisting of nonsingular functions of $\delta$ and $b$. In [3] this result is extended to show that in the case in which $\frac{n-m}{n}$ of a size $n$ ordered sample is censored from above, the first $m$ order statistics, $X_{(1,n)} \le X_{(2,n)} \le \cdots \le X_{(m,n)}$ are minimally sufficient so that for $m > 2$, the number of minimally sufficient statistics

exceeds the number of parameters to be estimated. Hence, there is no guarantee that when m is greater than 2 the maximum-likelihood estimators of Weibull parameters are the minimum-variance estimators of their expected values, as would be the case if the number of sufficient statistics equaled the number of parameters to be estimated (see Rao [4]). One cannot, therefore, claim the usual optimality properties for the maximum-likelihood estimators of the Weibull (or extreme-value) parameters when m is greater than two, but n is not large enough for asymptotic theory to apply. Moreover, it is not known when, for a given $p = m/n$, n is sufficiently large.

## LINEAR ESTIMATION

For estimating the general parametric function $\varphi = \ell_1 u + \ell_2 b$, the generalized Gauss-Markov Theorem (see Lloyd [5]) specifies the least-squares estimator as the unique best one among unbiased linear functions of $X_{(1,n)}, X_{(2,n)}, \ldots,$ and $X_{(m,n)}$ for all $m$ and $n$. Weibull estimates obtained graphically from probability plots are simply approximations to the least-squares estimate, with the major source of error being the necessarily subjective visual fitting of the least-squares line. Let $\varphi^* = \ell_1 u^* + \ell_2 b^*$ be the true least-squares or best linear unbiased (BLU) estimator of $\varphi$ based on the first $m$ of $n$ ordered sample observations, with $u^*$ and $b^*$ the BLU estimators of $u$ and $b$, respectively. These estimators ($u^*$, $b^*$, and $\varphi^*$) enjoy all the large sample properties attributed to maximum-likelihood estimators, including that of asymptotic normality (see Blom [6]).

By analogy with other distributions such as the normal, one might expect $u$, being a location parameter, to be estimated most efficiently by a linear function of the sufficient statistics, $X_{(1,n)}, X_{(2,n)}, \ldots, X_{(m,n)}$. For a scale parameter such as $b$, however, one might conjecture that considerably more efficient estimation could be achieved by some other means. That such a conjecture may well be unjustified is demonstrated by Table A-8B(6) in Dixon and Massey [7]. This table demonstrates the

extremely high efficiency of the best linear unbiased estimator of the Gaussian scale parameter $\sigma$ with respect to the uniformly minimum variance unbiased estimator which is nonlinear. The efficiency is 100% . $= m = 2$, is at its lowest (98.8%) for $n = m = 6$, and increases n for $n > 6$. This fact, along with results given in [8], supports author's conjecture that the best linear unbiased estimator of b has very high efficiency with respect to any unbiased estimator of b which is obtainable. The least-squares estimator $\varphi^*$ of $\varphi$ also has an advantage in that the estimate can be calculated directly as a linear function $\sum_{i=1}^{m} w_i X_{(i,n)}$ of weights $\{w_i\} = \ell_1 \{a_i\} + \ell_2\{c_i\}$, $i=1,2,\ldots m$, which can be determined from the first moments and second-moment matrix of the reduced order statistics, $Y_1, Y_2, \ldots, Y_m =$ $(X_{(1,n)}-u)/b$, $(X_{(2,n)}-u)/b, \ldots, (X_{(m,n)}-u)/b$ for any given m and n. The weights $\{a_i\}$ and $\{c_i\}$ allow for calculation of the BLU estimates of u and b respectively. The first- and second-moment matrices of the reduced order statistics and the weights for obtaining the estimates were determined in [3] for $2 \leq n \leq 25$, $2 \leq m \leq n$. The variances $\alpha b^2$ and $\gamma b^2$ of $u^*$ and $b^*$, respectively, and $\beta b^2$, the covariance of these estimators, were also calculated for the same values of m and n.

In [9], it is shown that for the class of linear estimators of $\varphi$ based on the first m of n extreme-value order statistics and with expected squared-error loss independent of u, there is a unique best

one given by $\tilde{\varphi} = \ell_1(u^*-[\beta/(1+\gamma)]b^* + [\ell_2/(1+\gamma)]b^* \equiv \ell_1\tilde{u} + \ell_2\tilde{b}$ , and with expected squared error equal to

$$\left[\ell_1^2\, E(LU) + 2\ell_1\ell_2\, E(CP) + \ell_2^2\, E(LB)\right]b^2 \equiv$$
$$\left\{\ell_1^2\alpha + 2\ell_1\ell_2\beta + \ell_2^2\gamma - [(\ell_1\beta + \ell_2\gamma)^2/(1+\gamma)]\right\}b^2 \quad , \tag{2}$$

for all $m$ and $n$. Let loss be defined as squared error divided by $b^2$. Then $\tilde{\varphi}$ is the best among linear estimators of $\varphi$ invariant under location and scalar transformations (the best linear invariant estimator) and has been shown in [10] to be the unique admissible minimax linear estimator of $\varphi$ based on $X_{(1,n)}, X_{(2,n)}, \ldots, X_{(m,n)}$ for all $m$ and $n$. It also has all the asymptotic properties of the BLU estimator plus that of asymptotic unbiasedness. Therefore, for the values of $m$ and $n$ for which tables of weights for obtaining the estimates based on the estimation rules $u^*$ and $b^*$ and tables of the covariance matrices of these estimators are available, weights $\{W_i\} = \ell_1 \{A_i\} + \ell_2 \{C_i\}$ for obtaining the BLI estimator $\tilde{\varphi}$ of $\varphi$, hence of $u$, $b$, or $x_R = u + b\log\log(1/R)$ , are easily obtainable. The expected error for $\tilde{u}$, $\tilde{b}$, or $\tilde{X}_R$ (the BLI estimator of $x_R$) can also be calculated from (2). The weights, $\{A_i\} = \{a_i-[\beta/(1+\gamma)]c_i\}$ and $\{C_i\} = \{c_i/(1+\gamma)\}$ , functions of $m$ and $n$, have been calculated and are given in Table C.I for $2 \le n \le 25$, $2 \le m \le n$. Values of E(LU) and E(LB), the expected losses of $u$ and $b$, respectively, and

$E(CP) \equiv E(\tilde{u}\tilde{b} - ub)/b^2$ were also calculated and appear in Table C.I. The indices $n$, $m$, and $i$ all appear in Table C.I as capital letters since no lower-case letters are available for the computer output. The method of utilizing the weights for estimating the parameters is illustrated in Figure 1. In this example, the ordering of the failure times and the calculation of the estimates has been achieved via a computer program which uses tapes upon which both the BLU and BLI weights are stored. It is evident, however, that a hand calculation of the estimates can be made with little difficulty.

EXAMPLE SHOWING METHOD OF COMPUTING PARAMETER ESTIMATES FROM FAILURE-TIME DATA

| FAILURE TIMES $n = 24$, $r = 24$ | | | WEIGHTS | |
|---|---|---|---|---|
| RANDOMIZED ORDER | ASCENDING ORDER | NATURAL LOGARITHMS | $A_i$ | $C_i$ |
| 119.0000 | 6.0000 | 1.7917595 | 0.0112481 | -0.02[illegible]931 |
| 138.0000 | 8.6000 | 2.1517622 | 0.0137390 | -0.0298709 |
| 146.0000 | 17.8000 | 2.8791985 | 0.0160213 | -0.0309171 |
| 151.0000 | 18.0000 | 2.8903718 | 0.0182155 | -0.0314605 |
| 27.5000 | 27.5000 | 3.3141860 | 0.0203699 | -0.03161[illegible]0 |
| 69.0000 | 33.5000 | 3.5115454 | 0.0225141 | -0.0314243 |
| 150.0000 | 50.5000 | 3.9219733 | 0.0246466 | -0.0309363 |
| 8.6000 | 51.5000 | 3.9415818 | 0.0268383 | -0.0300915 |
| 51.5000 | 69.0000 | 4.2341065 | 0.0290150 | -0.0289885 |
| 89.0000 | 74.0000 | 4.3040651 | 0.0313289 | -0.0274772 |
| 109.0000 | 74.0000 | 4.3040651 | 0.0335892 | -0.0256841 |
| 6.0000 | 89.0000 | 4.4886364 | 0.0359894 | -0.0234404 |
| 74.0000 | 109.0000 | 4.6913479 | 0.0385429 | -0.0207397 |
| 118.0000 | 118.0000 | 4.7706847 | 0.0411308 | -0.0175568 |
| 141.0000 | 119.0000 | 4.7791235 | 0.0439276 | -0.0137101 |
| 18.0000 | 138.0000 | 4.9272537 | 0.0468615 | -0.0091529 |
| 33.5000 | 141.0000 | 4.9487599 | 0.0500545 | -0.0036386 |
| 144.0000 | 144.0000 | 4.9698133 | 0.0535307 | 0.0030549 |
| 17.8000 | 146.0000 | 4.9836066 | 0.0573800 | 0.0113897 |
| 153.0000 | 150.0000 | 5.0106353 | 0.0617825 | 0.0219454 |
| 153.1000 | 151.0000 | 5.0172799 | 0.0669268 | 0.0360073 |
| 153.2000 | 153.0000 | 5.0304379 | 0.0733312 | 0.0559322 |
| 50.5000 | 153.1000 | 5.0310913 | 0.0821530 | 0.0882378 |
| 74.0000 | 153.2000 | 5.0317443 | 0.1008632 | 0.1981225 |
| | | | $\tilde{u} = 4.61398$ | $\tilde{b} = 0.56889$ |

SUMMARY

$\tilde{u} = 4.61398$ $\tilde{b} = 0.56889$

$\log\log(1/.90) = -2.25037$ $\log\log(1/.95) = -2.97020$

$\tilde{X}_{.90} = \tilde{u} + \tilde{b}\log\log(1/.90) = 3.33376$ $\tilde{T}_{.90} = 28.04366$

$\tilde{X}_{.95} = \tilde{u} + \tilde{b}\log\log(1/.95) = 2.92426$ $\tilde{T}_{.95} = 18.62038$

$\tilde{T}_R = \exp(\tilde{X}_R)$

Figure 1

## ACCURACY OF THE TABLES

The calculation of the moments of the order statistics, the original BLU weights $\{a_i\}$ and $\{c_i\}$, and the estimator covariance matrices were all based on derivations and results of Lieblein in [11] and [12]. All of the computer programs involved in making these calculations incorporated Rocketdyne's N-precision (somewhat more accurate than double precision) subroutines. Three constants, $\pi^2/6$, $\pi^2/12$, and Euler's constant, correct to 22 significant figures, were read in. All other preliminary values used in the computations were generated by using N-precision floating-point arithmetic. Differencing of the tabled values indicated that the numbers given were accurate to within a unit in the eighth decimal place. It should be noted, too, that the weights and the covariance matrices agreed precisely with those given by White for $2 \le n \le 20$ in [13] for the number of significant figures listed by him. In our computations the variances and covariances of the estimators were given by the values of the Lagrange multipliers specifying the constraints of invariance and unbiasedness for $u^*$ and $b^*$ as indicated in [9], and thus did not require a separate calculation. The BLI weights with the expected losses were obtained directly from the BLU weights and the variances and covariances of $u^*$ and $b^*$. Six decimal places only are given for the weights. The eight decimal places included for the expected losses imply additional accuracy for the weights.

ESTIMATION PROCEDURE FOR SAMPLES LARGER THAN 25

If n is greater than 25, the weights in Table C.I or the computer routine using these weights cannot be used directly. Estimates may be obtained, however, by randomly dividing the sample into an arbitrary number of sub-samples each of which has sample size less than 25. An estimate of b, u, or $x_R$ is then obtained as an average of the best linear unbiased sub-estimates where each is weighted by the reciprocal of the variance of the sub-estimator. An approximation to the best linear invariant estimate of each parameter can then be obtained from this average which is an unbiased linear estimate. If the total sample is divided into a set of k uncensored sub-samples of equal size, the Cramér-Rao efficiency of the estimate obtained by averaging is a function of the size $n_j$ of the sub-sample only, increasing with increasing $n_j$ (see Lieblein [12]). For more irregular partitionings of the sample, the determination of efficiency becomes more complex. For a fixed number k of sub-samples, however, the manner in which the sample is divided (e.g., two equal sub-samples or one large and one small) has little effect on efficiency. In any case, when the total sample is greater than 25, it can be seen from [8] that the Cramér-Rao efficiency of the estimate of reliable life will not usually fall below 87% even for R close to 1 unless there is a great deal of censoring.

Obtaining the proper unbiased estimate of $\ell_1 u + \ell_2 b$ (for each sub-sample) from which a weighted average may be obtained involves converting the estimates obtained from $\tilde{u}$ and $\tilde{b}$ to unbiased estimates. To accomplish this, we note that since

$\tilde{\varphi} = \ell_1 \tilde{u} + \ell_2 \tilde{b} = \ell_1 [u^* - (\beta/(1+\gamma))b^*] + [\ell_2/(1+\gamma)]b^*$, for all $\ell_1$ and $\ell_2$, then $b^* = (1+\gamma)\tilde{b}$ and $u^* = \tilde{u} + (\beta/(1+\gamma))b^* = \tilde{u} + E(CP)b^*$.

Furthermore, since $E(LB) = \gamma/(1+\gamma)$ and $E(CP) = \beta/(1+\gamma)$,

$\gamma = E(LB)/[1-E(LB)]$, $b^* = \tilde{b}/[1-E(LB)]$, and $\beta = E(CP)/[1-E(LB)]$. Thus

$u^* = \tilde{u} + [E(CP)/[1-E(LB)]]\tilde{b}$. The variance $\alpha$ of $u^*$ is

$[E(LU) + \beta^2/(1+\gamma)]b^2 = [E(LU) + [E(CP)]^2/[1-E(LB)]]b^2$. We let $k$ be the total number of sub-samples and we let $Q_j$ be equal to

$\ell_1^2 \alpha_j + 2\ell_1 \ell_2 \beta_j + + \ell_2^2 \gamma_j, j = 1, 2, \ldots, k.$

Then the variance of the unbiased estimator formed by the weighted average,

$$\bar{\varphi} = \frac{1}{\sum_{j=1}^{k} (1/Q_j)} \sum_{j=1}^{k} (1/Q_j)(\ell_1 u_j^* + \ell_2 b_j^*)$$ is

$$\frac{b^2}{\left[\sum_{j=1}^{k} (1/Q_j)\right]^2} \sum_{j=1}^{k} (1/Q_j)^2 (\ell_1^2 \alpha_j + 2\ell_1\ell_2\beta_j + \ell_2^2\gamma_j) = \frac{b^2}{\sum_{j=1}^{k} (1/Q_j)^2},$$ where

the subscript $j$ indicates with which of the sub-samples a quantity is associated. It is simple to demonstrate that $\bar{\varphi}$ has the minimum variance among unbiased estimators based on the sub-estimators based on the given sub-samples.

The estimator $\varphi' = \bar{\varphi} - \bar{\beta}\,\bar{b}/(1 + \sum_{j=1}^{k}(1/\gamma_j))$ (where $\bar{b} = \frac{1}{\sum_{j=1}^{k}\gamma_j}\sum_{j=1}^{k}(1/\gamma_j)b_j^*$ is the minimum-variance unbiased estimator of b based on the given sub-estimators $\bar{\beta} = \frac{1}{\sum_{j=1}^{k}(1/Q_j)\sum_{j=1}^{k}(1/\gamma_j)}\sum_{j=1}^{k}(1/Q_j)(1/\gamma_j)(\ell_1\beta_j + \ell_2\gamma_j)$ is the covariance between $\bar{\varphi}$ and $\bar{b}$) is the invariant linear function of $\bar{\varphi}$ and $\bar{b}$ with smallest expected squared deviation from $\varphi$. It can be shown, by using methods very similar to those used in [8] to derive the form of the Cramér-Rao bound for invariant estimators of location parameters, that $\varphi'$ is the best linear invariant estimator of $\varphi$ based on the given sub-estimators.

An alternative method of handling samples larger than 25 has been derived by McCool and is explained in [14]. For large m, however, this method requires considerable additional compution until such time as additional tables combining McCool's results with those in Table C.I are made available. Another approach is described by Johns and Lieberman in [15]. The estimates which can be obtained from tables of weights given by Johns and Lieberman are good approximations to the best linear invariant estimates of u and b. These weights are given for n equal to 10, 15, 20, 30, 50, and 100 and four values of m for each n.

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## TABLE C.I

WEIGHTS FOR OBTAINING BEST LINEAR INVARIANT ESTIMATES

OF PARAMETERS OF THE EXTREME-VALUE DISTRIBUTION

A(N,M,I) = WEIGHT FOR ESTIMATING U
C(N,M,I) = WEIGHT FOR ESTIMATING B
E(LU) = EXPECTED LOSS FOR ESTIMATE OF U
E(CP) = EXPECTED CROSS PRODUCT
E(LB) = EXPECTED LOSS FOR ESTIMATE OF B

| | N | M | I | A(N,M,I) | | C(N,M,I) |
|---|---|---|---|---|---|---|
| | 2 | 2 | 1 | 0.110731 | | -0.421383 |
| | | | 2 | 0.889269 | | 0.421383 |
| E(LU) | | | | 0.65712995 | | |
| E(CP) | | | | | 0.03757418 | |
| E(LB) | | | | | | 0.41583918 |
| | 3 | 2 | 1 | -0.166001 | | -0.452110 |
| | | | 2 | 1.166001 | | 0.452110 |
| E(LU) | | | | 0.79546061 | | |
| E(CP) | | | | | 0.25750956 | |
| E(LB) | | | | | | 0.45005549 |
| | 3 | 3 | 1 | 0.081063 | | -0.278666 |
| | | | 2 | 0.251001 | | -0.190239 |
| | | | 3 | 0.667936 | | 0.468904 |
| E(LU) | | | | 0.40240741 | | |
| E(CP) | | | | | -0.01842169 | |
| E(LB) | | | | | | 0.25634620 |
| | 4 | 2 | 1 | -0.346974 | | -0.465455 |
| | | | 2 | 1.346974 | | 0.465455 |
| E(LU) | | | | 1.01477788 | | |
| E(CP) | | | | | 0.41350875 | |
| E(LB) | | | | | | 0.46438768 |
| | 4 | 3 | 1 | -0.044975 | | -0.297651 |
| | | | 2 | 0.088057 | | -0.234054 |
| | | | 3 | 0.956918 | | 0.531705 |
| E(LU) | | | | 0.42315147 | | |
| E(CP) | | | | | 0.08477554 | |
| E(LB) | | | | | | 0.28172930 |
| | 4 | 4 | 1 | 0.064336 | | -0.203052 |
| | | | 2 | 0.147340 | | -0.182749 |
| | | | 3 | 0.261510 | | -0.070109 |
| | | | 4 | 0.526813 | | 0.455910 |
| E(LU) | | | | 0.29247651 | | |
| E(CP) | | | | | -0.02831210 | |
| E(LB) | | | | | | 0.18386193 |
| | 5 | 2 | 1 | -0.481434 | | -0.472962 |
| | | | 2 | 1.481434 | | 0.472962 |
| E(LU) | | | | 1.24921018 | | |
| E(CP) | | | | | 0.53379141 | |
| E(LB) | | | | | | 0.47230837 |
| | 5 | 3 | 1 | -0.137958 | | -0.306562 |
| | | | 2 | -0.025510 | | -0.257087 |
| | | | 3 | 1.163468 | | 0.563650 |
| E(LU) | | | | 0.49029288 | | |
| E(CP) | | | | | 0.16612899 | |
| E(LB) | | | | | | 0.29419192 |
| | 5 | 4 | 1 | -0.006983 | | -0.217766 |
| | | | 2 | 0.059652 | | -0.199351 |
| | | | 3 | 0.156664 | | -0.118927 |
| | | | 4 | 0.790668 | | 0.536044 |
| E(LU) | | | | 0.29062766 | | |
| E(CP) | | | | | 0.03076329 | |
| E(LB) | | | | | | 0.202418[illegible] |
| | 5 | 5 | 1 | 0.052975 | | -0.158131 |
| | | | 2 | 0.103531 | | -0.155707 |
| | | | 3 | 0.163808 | | -0.111820 |
| | | | 4 | 0.246092 | | -0.005600 |
| | | | 5 | 0.433593 | | 0.431259 |
| E(LU) | | | | 0.23040495 | | |
| E(CP) | | | | | -0.02913523 | |
| E(LB) | | | | | | 0.14284288 |
| | 6 | 2 | 1 | -0.588298 | | -0.477782 |
| | | | 2 | 1.588298 | | 0.477782 |
| E(LU) | | | | 1.48102383 | | |
| E(CP) | | | | | 0.63148980 | |
| E(LB) | | | | | | 0.47734078 |
| | 6 | 3 | 1 | -0.211474 | | -0.311047 |
| | | | 2 | -0.112994 | | -0.271381 |
| | | | 3 | 1.324468 | | 0.583229 |
| E(LU) | | | | 0.57539484 | | |
| E(CP) | | | | | 0.23269670 | |
| E(LB) | | | | | | 0.30173252 |

TABLE OF WEIGHTS (CONTINUED)

| | N | M | I | A(N,M,I) | | C(N,M,I) |
|---|---|---|---|---|---|---|
| | 6 | 4 | 1 | -0.063569 | | -0.225141 |
| | | | 2 | -0.006726 | | -0.209083 |
| | | | 3 | 0.079882 | | -0.146386 |
| | | | 4 | 0.990412 | | 0.580610 |
| E(LU) | | | | 0.31552097 | | |
| E(CP) | | | | | 0.08035062 | |
| E(LB) | | | | | | 0.21242254 |
| | 6 | 5 | 1 | 0.007521 | | -0.169920 |
| | | | 2 | 0.048328 | | -0.166319 |
| | | | 3 | 0.101608 | | -0.129510 |
| | | | 4 | 0.172859 | | -0.054453 |
| | | | 5 | 0.669685 | | 0.520201 |
| E(LU) | | | | 0.22351297 | | |
| E(CP) | | | | | 0.00888019 | |
| E(LB) | | | | | | 0.15690540 |
| | 6 | 6 | 1 | 0.044826 | | -0.128810 |
| | | | 2 | 0.079377 | | -0.132102 |
| | | | 3 | 0.117541 | | -0.111951 |
| | | | 4 | 0.163591 | | -0.064666 |
| | | | 5 | 0.226486 | | 0.031796 |
| | | | 6 | 0.368179 | | 0.405733 |
| E(LU) | | | | 0.19030430 | | |
| E(CP) | | | | | -0.02771574 | |
| E(LB) | | | | | | 0.11657671 |
| | 7 | 2 | 1 | -0.676894 | | -0.481140 |
| | | | 2 | 1.676894 | | 0.481140 |
| E(LU) | | | | 1.70468001 | | |
| E(CP) | | | | | 0.71366553 | |
| E(LB) | | | | | | 0.48082310 |
| | 7 | 3 | 1 | -0.272195 | | -0.315369 |
| | | | 2 | -0.184061 | | -0.281139 |
| | | | 3 | 1.456255 | | 0.596507 |
| E(LU) | | | | 0.66758707 | | |
| E(CP) | | | | | 0.28885432 | |
| E(LB) | | | | | | 0.30681307 |
| | 7 | 4 | 1 | -0.110274 | | -0.229691 |
| | | | 2 | -0.060226 | | -0.215613 |
| | | | 3 | 0.018671 | | -0.164168 |
| | | | 4 | 1.151829 | | 0.609472 |
| E(LU) | | | | 0.35340223 | | |
| E(CP) | | | | | 0.12260834 | |
| E(LB) | | | | | | 0.21884662 |
| | 7 | 5 | 1 | -0.030368 | | -0.176203 |
| | | | 2 | 0.004333 | | -0.172399 |
| | | | 3 | 0.052957 | | -0.141218 |
| | | | 4 | 0.117599 | | -0.082820 |
| | | | 5 | 0.855480 | | 0.572640 |
| E(LU) | | | | 0.23316740 | | |
| E(CP) | | | | | 0.04212562 | |
| E(LB) | | | | | | 0.16497315 |
| | 7 | 6 | 1 | 0.013524 | | -0.138436 |
| | | | 2 | 0.041588 | | -0.140342 |
| | | | 3 | 0.075499 | | -0.121821 |
| | | | 4 | 0.117461 | | -0.082938 |
| | | | 5 | 0.172092 | | -0.015394 |
| | | | 6 | 0.579635 | | 0.498931 |
| E(LU) | | | | 0.18269947 | | |
| E(CP) | | | | | -0.00130057 | |
| E(LB) | | | | | | 0.12760617 |
| | 7 | 7 | 1 | 0.038743 | | -0.108323 |
| | | | 2 | 0.064086 | | -0.113479 |
| | | | 3 | 0.090785 | | -0.103569 |
| | | | 4 | 0.120971 | | -0.078748 |
| | | | 5 | 0.157657 | | -0.032632 |
| | | | 6 | 0.207825 | | 0.054727 |
| | | | 7 | 0.31993[illegible] | | 0.382022 |
| E(LU) | | | | 0.16219070 | | |
| E(CP) | | | | | -0.02578937 | |
| E(LB) | | | | | | 0.09836496 |
| | 8 | 2 | 1 | -0.752513 | | -0.483616 |
| | | | 2 | 1.752513 | | 0.483616 |
| E(LU) | | | | 1.91861540 | | |
| E(CP) | | | | | 0.78453314 | |
| E(LB) | | | | | | 0.48337662 |
| | 8 | 3 | 1 | -0.323875 | | -0.317890 |
| | | | 2 | -0.243808 | | -0.288231 |
| | | | 3 | 1.567683 | | 0.606120 |
| E(LU) | | | | 0.76198737 | | |
| E(CP) | | | | | 0.33734068 | |
| E(LB) | | | | | | 0.31047652 |
| | 8 | 4 | 1 | -0.149973 | | -0.232805 |
| | | | 2 | -0.105015 | | -0.220324 |
| | | | 3 | -0.032257 | | -0.176675 |
| | | | 4 | 1.287245 | | 0.629805 |
| E(LU) | | | | 0.39805551 | | |
| E(CP) | | | | | 0.15928131 | |
| E(LB) | | | | | | 0.22335819 |
| | 8 | 5 | 1 | -0.062656 | | -0.180231 |
| | | | 2 | -0.032248 | | -0.176510 |
| | | | 3 | 0.012767 | | -0.149566 |
| | | | 4 | 0.072446 | | -0.101642 |
| | | | 5 | 1.009691 | | 0.607948 |
| E(LU) | | | | 0.25192092 | | |
| E(CP) | | | | | 0.07129172 | |
| E(LB) | | | | | | 0.17037848 |
| | 8 | 6 | 1 | -0.013509 | | -0.143834 |
| | | | 2 | 0.010292 | | -0.145006 |
| | | | 3 | 0.041357 | | -0.128393 |
| | | | 4 | 0.080475 | | -0.095696 |
| | | | 5 | 0.130327 | | -0.043280 |
| | | | 6 | 0.751058 | | 0.556209 |
| E(LU) | | | | 0.18599844 | | |
| E(CP) | | | | | 0.02247163 | |
| E(LB) | | | | | | 0.13422386 |

TABLE OF WEIGHTS (CONTINUED)

| | N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|---|
| | 8 | 7 | 1 | 0.015973 | -0.116317 |
| | | | 2 | 0.036729 | -0.120331 |
| | | | 3 | 0.060439 | -0.110582 |
| | | | 4 | 0.088239 | -0.088450 |
| | | | 5 | 0.122062 | -0.050995 |
| | | | 6 | 0.165529 | 0.009700 |
| | | | 7 | 0.511030 | 0.476975 |
| E(LU) | | | | 0.15505149 | |
| E(CP) | | | | | -0.00641304 |
| E(LB) | | | | | 0.10726405 |
| | 8 | 8 | 1 | 0.034052 | -0.093270 |
| | | | 2 | 0.053552 | -0.098886 |
| | | | 3 | 0.073452 | -0.093994 |
| | | | 4 | 0.095062 | -0.079752 |
| | | | 5 | 0.119768 | -0.053918 |
| | | | 6 | 0.149934 | -0.010179 |
| | | | 7 | 0.191236 | 0.069325 |
| | | | 8 | 0.282943 | 0.360675 |
| E(LU) | | | | 0.14136026 | |
| E(CP) | | | | | -0.02386561 |
| E(LB) | | | | | 0.08501680 |
| | 9 | 2 | 1 | -0.818444 | -0.485517 |
| | | | 2 | 1.818444 | 0.485517 |
| E(LU) | | | | 2.12272209 | |
| E(CP) | | | | | 0.84680378 |
| E(LB) | | | | | 0.48532951 |
| | 9 | 3 | 1 | -0.368833 | -0.319786 |
| | | | 2 | -0.295280 | -0.293621 |
| | | | 3 | 1.664113 | 0.613407 |
| E(LU) | | | | 0.85621748 | |
| E(CP) | | | | | 0.37995861 |
| E(LB) | | | | | 0.31324611 |
| | 9 | 4 | 1 | -0.184461 | -0.235080 |
| | | | 2 | -0.143505 | -0.223891 |
| | | | 3 | -0.075815 | -0.185970 |
| | | | 4 | 1.403781 | 0.644941 |
| E(LU) | | | | 0.44625568 | |
| E(CP) | | | | | 0.19160927 |
| E(LB) | | | | | 0.22671251 |
| | 9 | 5 | 1 | -0.090726 | -0.183061 |
| | | | 2 | -0.063541 | -0.179515 |
| | | | 3 | -0.021495 | -0.155825 |
| | | | 4 | 0.034159 | -0.115133 |
| | | | 5 | 1.141604 | 0.633534 |
| E(LU) | | | | 0.27605014 | |
| E(CP) | | | | | 0.09715351 |
| E(LB) | | | | | 0.17429417 |
| | 9 | 6 | 1 | -0.037118 | -0.147411 |
| | | | 2 | -0.016377 | -0.148150 |
| | | | 3 | 0.012499 | -0.133219 |
| | | | 4 | 0.049305 | -0.105060 |
| | | | 5 | 0.095814 | -0.062073 |
| | | | 6 | 0.896078 | 0.595913 |
| E(LU) | | | | 0.19579592 | |
| E(CP) | | | | | 0.04378261 |
| E(LB) | | | | | 0.13880129 |
| | 9 | 7 | 1 | -0.004220 | -0.120988 |
| | | | 2 | 0.013386 | -0.124245 |
| | | | 3 | 0.035068 | -0.115091 |
| | | | 4 | 0.061198 | -0.095508 |
| | | | 5 | 0.093013 | -0.064162 |
| | | | 6 | 0.132740 | -0.017187 |
| | | | 7 | 0.668815 | 0.537180 |
| E(LU) | | | | 0.15547192 | |
| E(CP) | | | | | 0.01139509 |
| E(LB) | | | | | 0.11278822 |
| | 9 | 8 | 1 | 0.016797 | -0.100011 |
| | | | 2 | 0.032919 | -0.104750 |
| | | | 3 | 0.050582 | -0.095608 |
| | | | 4 | 0.070497 | -0.086226 |
| | | | 5 | 0.093635 | -0.063541 |
| | | | 6 | 0.121560 | -0.028346 |
| | | | 7 | 0.157175 | 0.026525 |
| | | | 8 | 0.456836 | 0.455956 |
| E(LU) | | | | 0.13496842 | |
| E(CP) | | | | | -0.00906894 |
| E(LB) | | | | | 0.09236358 |
| | 9 | 9 | 1 | 0.030338 | -0.081777 |
| | | | 2 | 0.045872 | -0.087308 |
| | | | 3 | 0.061368 | -0.085084 |
| | | | 4 | 0.077742 | -0.076470 |
| | | | 5 | 0.095769 | -0.060667 |
| | | | 6 | 0.116517 | -0.035136 |
| | | | 7 | 0.141932 | 0.006001 |
| | | | 8 | 0.176764 | 0.078828 |
| | | | 9 | 0.253697 | 0.341614 |
| E(LU) | | | | 0.12529518 | |
| E(CP) | | | | | -0.02209438 |
| E(LB) | | | | | 0.07482425 |
| | 10 | 2 | 1 | -0.876869 | -0.487022 |
| | | | 2 | 1.876869 | 0.487022 |
| E(LU) | | | | 2.31744054 | |
| E(CP) | | | | | 0.90232208 |
| E(LB) | | | | | 0.48687150 |
| | 10 | 3 | 1 | -0.408602 | -0.321265 |
| | | | 2 | -0.340443 | -0.297858 |
| | | | 3 | 1.749045 | 0.619124 |
| E(LU) | | | | 0.94907551 | |
| E(CP) | | | | | 0.41795081 |
| E(LB) | | | | | 0.31541467 |
| | 10 | 4 | 1 | -0.214930 | -0.236817 |
| | | | 2 | -0.177223 | -0.226688 |
| | | | 3 | -0.113820 | -0.193159 |
| | | | 4 | 1.505973 | 0.656663 |
| E(LU) | | | | 0.49619736 | |
| E(CP) | | | | | 0.22047816 |
| E(LB) | | | | | 0.22930885 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 10 | 5 | 1 | -0.115524 | -0.185169 |
| | | 2 | -0.090868 | -0.181821 |
| | | 3 | -0.051341 | -0.160697 |
| | | 4 | 0.0009[illegible] | -0.125311 |
| | | 5 | 1.25[illegible] | 0.652997 |
| E(LU) | | | 0.30[illegible]4549 | |
| E(CP) | | | 0.12[illegible]3305 | |
| E(LB) | | | | 0.[illegible]7727542 |
| 10 | 6 | 1 | -0.058017 | -0.149985 |
| | | 2 | -0.039595 | -0.150451 |
| | | 3 | -0.012513 | -0.136941 |
| | | 4 | 0.022314 | -0.112224 |
| | | 5 | 0.065750 | -0.075721 |
| | | 6 | 1.022062 | 0.625321 |
| E(LU) | | | 0.20973843 | |
| E(CP) | | | 0.06299841 | |
| E(LB) | | | | 0.14219828 |
| 10 | 7 | 1 | -0.022198 | -0.124170 |
| | | 2 | -0.006909 | -0.126894 |
| | | 3 | 0.013224 | -0.118392 |
| | | 4 | 0.037994 | -0.100924 |
| | | 5 | 0.068153 | -0.073988 |
| | | 6 | 0.105164 | -0.035501 |
| | | 7 | 0.804572 | 0.579868 |
| E(LU) | | | 0.16066059 | |
| E(CP) | | | 0.02762724 | |
| E(LB) | | | | 0.11670571 |
| 10 | 8 | 1 | 0.001179 | -0.104082 |
| | | 2 | 0.014889 | -0.108163 |
| | | 3 | 0.030998 | -0.103119 |
| | | 4 | 0.049734 | -0.090835 |
| | | 5 | 0.071745 | -0.070902 |
| | | 6 | 0.098114 | -0.041560 |
| | | 7 | 0.130649 | 0.000799 |
| | | 8 | 0.602692 | 0.517864 |
| E(LU) | | | 0.13403554 | |
| E(CP) | | | 0.00474963 | |
| E(LB) | | | | 0.09704810 |
| 10 | 9 | 1 | 0.016841 | -0.087538 |
| | | 2 | 0.029807 | -0.092405 |
| | | 3 | 0.043570 | -0.089839 |
| | | 4 | 0.058640 | -0.081428 |
| | | 5 | 0.075576 | -0.066855 |
| | | 6 | 0.095169 | -0.044670 |
| | | 7 | 0.118707 | -0.011816 |
| | | 8 | 0.148575 | 0.038159 |
| | | 9 | 0.413116 | 0.436394 |
| E(LU) | | | 0.11965747 | |
| E(CP) | | | -0.01043859 | |
| E(LB) | | | | 0.08100409 |
| 10 | 10 | 1 | 0.027331 | -0.072734 |
| | | 2 | 0.040034 | -0.077971 |
| | | 3 | 0.052496 | -0.077242 |
| | | 4 | 0.065408 | -0.071876 |
| | | 5 | 0.079263 | -0.061652 |
| | | 6 | 0.094638 | -0.045420 |
| | | 7 | 0.112414 | -0.020698 |
| | | 8 | 0.134239 | 0.017927 |
| | | 9 | 0.164179 | 0.085070 |
| | | 10 | 0.230001 | 0.324597 |
| E(LU) | | | 0.11252220 | |
| E(CP) | | | -0.02050852 | |
| E(LB) | | | | 0.06679250 |
| 11 | 2 | 1 | -0.929310 | -0.488243 |
| | | 2 | 1.929310 | 0.488243 |
| E(LU) | | | 2.50340024 | |
| E(CP) | | | 0.95239887 | |
| E(LB) | | | | 0.48812000 |
| 11 | 3 | 1 | -0.444245 | -0.322452 |
| | | 2 | -0.380642 | -0.301277 |
| | | 3 | 1.824887 | 0.62[illegible]29 |
| E(LU) | | | 1.03995578 | |
| E(CP) | | | 0.45220741 | |
| E(LB) | | | | 0.3171[illegible]30 |
| 11 | 4 | 1 | -0.242206 | -0.238188 |
| | | 2 | -0.207204 | -0.228941 |
| | | 3 | -0.147490 | -0.198888 |
| | | 4 | 1.596900 | 0.666017 |
| E(LU) | | | 0.54681985 | |
| E(CP) | | | 0.24633583 | |
| E(LB) | | | | 0.23138012 |
| 11 | 5 | 1 | -0.137718 | -0.186803 |
| | | 2 | -0.115110 | -0.183651 |
| | | 3 | -0.077762 | -0.164597 |
| | | 4 | -0.028411 | -0.133278 |
| | | 5 | 1.359000 | 0.668329 |
| E(LU) | | | 0.33282848 | |
| E(CP) | | | 0.14129911 | |
| E(LB) | | | | 0.17962678 |
| 11 | 6 | 1 | -0.076739 | -0.151936 |
| | | 2 | -0.060142 | -0.152221 |
| | | 3 | -0.034581 | -0.139907 |
| | | 4 | -0.001490 | -0.117886 |
| | | 5 | 0.03951[illegible] | -0.086131 |
| | | 6 | 1.13343[illegible] | 0.648081 |
| E(LU) | | | 0.22640907 | |
| E(CP) | | | 0.08045010 | |
| E(LB) | | | | 0.14483423 |
| 11 | 7 | 1 | -0.038349 | -0.126507 |
| | | 2 | -0.024842 | -0.128838 |
| | | 3 | -0.005964 | -0.120951 |
| | | 4 | 0.017632 | -0.105219 |
| | | [illegible] | 0.046354 | -0.081602 |
| | | 6 | 0.061182 | -0.048929 |
| | | 7 | 0.923987 | 0.612047 |
| E(LU) | | | 0.16905710 | |
| E(CP) | | | 0.04246025 | |
| E(LB) | | | | 0.11966982 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 11 | 8 | 1 | -0.012943 | -0.106922 |
| | | 2 | -0.001050 | -0.110498 |
| | | 3 | 0.013869 | -0.105662 |
| | | 4 | 0.031661 | -0.094405 |
| | | 5 | 0.052723 | -0.076693 |
| | | 6 | 0.077815 | -0.051525 |
| | | 7 | 0.108161 | -0.016860 |
| | | 8 | 0.729765 | 0.562564 |
| E(LU) | | | 0.13669382 | |
| E(CP) | | | 0.01751192 | |
| E(LB) | | | | 0.10043756 |
| 11 | 9 | 1 | 0.004425 | -0.091115 |
| | | 2 | 0.015498 | -0.095437 |
| | | 3 | 0.028023 | -0.092780 |
| | | 4 | 0.042178 | -0.084833 |
| | | 5 | 0.058340 | -0.071581 |
| | | 6 | 0.077093 | -0.052182 |
| | | 7 | 0.099349 | -0.024880 |
| | | 8 | 0.126592 | 0.013606 |
| | | 9 | 0.548502 | 0.499201 |
| E(LU) | | | 0.11809425 | |
| E(CP) | | | 0.00058414 | |
| E(LB) | | | | 0.08503131 |
| 11 | 10 | 1 | 0.016502 | -0.077717 |
| | | 2 | 0.027205 | -0.082449 |
| | | 3 | 0.038291 | -0.081388 |
| | | 4 | 0.050160 | -0.075977 |
| | | 5 | 0.063170 | -0.066222 |
| | | 6 | 0.077772 | -0.051429 |
| | | 7 | 0.094625 | -0.030120 |
| | | 8 | 0.114811 | 0.000537 |
| | | 9 | 0.140333 | 0.046381 |
| | | 10 | 0.377130 | 0.418384 |
| E(LU) | | | 0.10756449 | |
| E(CP) | | | -0.01109747 | |
| E(LB) | | | | 0.07207183 |
| 11 | 11 | 1 | 0.024850 | -0.065444 |
| | | 2 | 0.035456 | -0.070318 |
| | | 3 | 0.045727 | -0.070456 |
| | | 4 | 0.056215 | -0.067076 |
| | | 5 | 0.067261 | -0.060207 |
| | | 6 | 0.079220 | -0.049300 |
| | | 7 | 0.092560 | -0.033156 |
| | | 8 | 0.108034 | -0.009427 |
| | | 9 | 0.127068 | 0.026879 |
| | | 10 | 0.153197 | 0.089148 |
| | | 11 | 0.210412 | 0.309357 |
| E(LU) | | | 0.10212039 | |
| E(CP) | | | -0.01910164 | |
| E(LB) | | | | 0.06030372 |
| 12 | 2 | 1 | -0.976872 | -0.489254 |
| | | 2 | 1.976872 | 0.489254 |
| E(LU) | | | 2.68127021 | |
| E(CP) | | | 0.99799849 | |
| E(LB) | | | | 0.48915157 |
| 12 | 3 | 1 | -0.476530 | -0.323426 |
| | | 2 | -0.416836 | -0.304093 |
| | | 3 | 1.893367 | 0.627519 |
| E(LU) | | | 1.12857097 | |
| E(CP) | | | 0.48338667 | |
| E(LB) | | | | 0.31859354 |
| 12 | 4 | 1 | -0.266888 | -0.239300 |
| | | 2 | -0.234180 | -0.230796 |
| | | 3 | -0.177681 | -0.203562 |
| | | 4 | 1.678749 | 0.673657 |
| E(LU) | | | 0.59748043 | |
| E(CP) | | | 0.27026774 | |
| E(LB) | | | | 0.23307201 |
| 12 | 5 | 1 | -0.157792 | -0.188109 |
| | | 2 | -0.136884 | -0.185142 |
| | | 3 | -0.101445 | -0.167790 |
| | | 4 | -0.054640 | -0.139693 |
| | | 5 | 1.450761 | 0.680734 |
| E(LU) | | | 0.36338878 | |
| E(CP) | | | 0.16042600 | |
| E(LB) | | | | 0.18153147 |
| 12 | 6 | 1 | -0.093679 | -0.153471 |
| | | 2 | -0.078561 | -0.153632 |
| | | 3 | -0.054320 | -0.142329 |
| | | 4 | -0.022769 | -0.122474 |
| | | 5 | 0.016136 | -0.094355 |
| | | 6 | 1.233193 | 0.666261 |
| E(LU) | | | 0.24490094 | |
| E(CP) | | | 0.09641022 | |
| E(LB) | | | | 0.14694548 |
| 12 | 7 | 1 | -0.052987 | -0.128308 |
| | | 2 | -0.040893 | -0.130339 |
| | | 3 | -0.023072 | -0.123007 |
| | | 4 | -0.000515 | -0.109712 |
| | | 5 | 0.026930 | -0.087681 |
| | | 6 | 0.059918 | -0.059256 |
| | | 7 | 1.030620 | 0.637304 |
| E(LU) | | | 0.17967935 | |
| E(CP) | | | 0.05607919 | |
| E(LB) | | | | 0.12200601 |
| 12 | 8 | 1 | -0.025785 | -0.109045 |
| | | 2 | -0.015312 | -0.112224 |
| | | 3 | -0.001353 | -0.107627 |
| | | 4 | 0.015634 | -0.097276 |
| | | 5 | 0.035853 | -0.081361 |
| | | 6 | 0.059835 | -0.059315 |
| | | 7 | 0.088444 | -0.029900 |
| | | 8 | 0.842684 | 0.596748 |
| E(LU) | | | 0.14186580 | |
| E(CP) | | | 0.02930146 | |
| E(LB) | | | | 0.10304331 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 12 | 9 | 1 | -0.006944 | -0.093658 |
| | | 2 | 0.002669 | -0.097540 |
| | | 3 | 0.014239 | -0.094893 |
| | | 4 | 0.027669 | -0.087448 |
| | | 5 | 0.043189 | -0.075371 |
| | | 6 | 0.061225 | -0.058180 |
| | | 7 | 0.082441 | -0.034802 |
| | | 8 | 0.107856 | -0.003342 |
| | | 9 | 0.667655 | 0.545234 |
| E(LU) | | | 0.11929957 | |
| E(CP) | | | 0.01087297 | |
| E(LB) | | | | 0.08799386 |
| 12 | 10 | 1 | 0.006411 | -0.080881 |
| | | 2 | 0.015598 | -0.085171 |
| | | 3 | 0.025675 | -0.083952 |
| | | 4 | 0.036799 | -0.078714 |
| | | 5 | 0.049211 | -0.069610 |
| | | 6 | 0.063256 | -0.056237 |
| | | 7 | 0.0794[illegible] | -0.037675 |
| | | 8 | 0.0985[illegible] | -0.012272 |
| | | 9 | 0.121752 | 0.022956 |
| | | 10 | 0.503338 | 0.481555 |
| E(LU) | | | 0.10573191 | |
| E(CP) | | | -0.00210755 | |
| E(LB) | | | | 0.07557509 |
| 12 | 11 | 1 | 0.01[illegible] | -0.069798 |
| | | 2 | 0.0249[illegible]7 | -0.074285 |
| | | 3 | 0.0341[illegible] | -0.074131 |
| | | 4 | 0.043790 | -0.070617 |
| | | 5 | 0.054149 | -0.063891 |
| | | 6 | 0.065515 | -0.053621 |
| | | 7 | 0.078264 | -0.039034 |
| | | 8 | 0.097958 | -0.018715 |
| | | 9 | 0.11052[illegible] | 0.009948 |
| | | 10 | 0.137666 | 0.052280 |
| | | 11 | 0.347033 | 0.401864 |
| E(LU) | | | 0.09775217 | |
| E(CP) | | | -0.01134890 | |
| E(LB) | | | | 0.06487266 |
| 12 | 12 | 1 | 0.022771 | -0.059449 |
| | | 2 | 0.031776 | -0.063952 |
| | | 3 | 0.040408 | -0.064601 |
| | | 4 | 0.049122 | -0.062489 |
| | | 5 | 0.058175 | -0.057754 |
| | | 6 | 0.067800 | -0.050137 |
| | | 7 | 0.078281 | -0.039010 |
| | | 8 | 0.090017 | -0.023199 |
| | | 9 | 0.103664 | -0.000505 |
| | | 10 | 0.120475 | 0.033696 |
| | | 11 | 0.143566 | 0.091751 |
| | | 12 | 0.193[illegible]47 | 0.295648 |
| E(LU) | | | 0.09348388 | |
| E(CP) | | | -0.01785537 | |
| E(LB) | | | | 0.05495436 |
| 13 | 2 | 1 | -1.020378 | -0.490105 |
| | | 2 | 1.020377 | 0.490105 |
| E(LU) | | | 2.85169694 | |
| E(CP) | | | 1.03985071 | |
| E(LB) | | | | 0.49001823 |
| 13 | 3 | 1 | -0.506031 | -0.324239 |
| | | 2 | -0.449735 | -0.306454 |
| | | 3 | 1.955765 | 0.630694 |
| E(LU) | | | 1.21480934 | |
| E(CP) | | | 0.51198847 | |
| E(LB) | | | | 0.31979363 |
| 13 | 4 | 1 | -0.289420 | -0.240219 |
| | | 2 | -0.258687 | -0.232349 |
| | | 3 | -0.205024 | -0.207450 |
| | | 4 | 1.753131 | 0.680016 |
| E(LU) | | | 0.64778295 | |
| E(CP) | | | 0.29204583 | |
| E(LB) | | | | 0.23448055 |
| 13 | 5 | 1 | -0.176109 | -0.189177 |
| | | 2 | -0.156637 | -0.186381 |
| | | 3 | -0.122893 | -0.170454 |
| | | 4 | -0.078337 | -0.144971 |
| | | 5 | 1.533976 | 0.690983 |
| E(LU) | | | 0.39459617 | |
| E(CP) | | | 0.1.799724 | |
| E(LB) | | | | 0.18310709 |
| 13 | 6 | 1 | -0.109140 | -0.154711 |
| | | 2 | -0.095246 | -0.154785 |
| | | 3 | -0.072165 | -0.144347 |
| | | 4 | -0.041997 | -0.126268 |
| | | 5 | -0.004940 | -0.101028 |
| | | 6 | 1.323488 | 0.681140 |
| E(LU) | | | 0.26460952 | |
| E(CP) | | | 0.11109896 | |
| E(LB) | | | | 0.14867755 |
| 13 | 7 | 1 | -0.066358 | -0.129743 |
| | | 2 | -0.055414 | -0.131538 |
| | | 3 | -0.038503 | -0.124701 |
| | | 4 | -0.016879 | -0.111609 |
| | | 5 | 0.009416 | -0.092649 |
| | | 6 | 0.040810 | -0.067475 |
| | | 7 | 1.126930 | 0.657714 |
| E(LU) | | | 0.19187273 | |
| E(CP) | | | 0.06864731 | |
| E(LB) | | | | 0.12390133 |
| 13 | 8 | 1 | -0.037540 | -0.110704 |
| | | 2 | -0.028206 | -0.113563 |
| | | 3 | -0.015049 | -0.109206 |
| | | 4 | 0.001231 | -0.099644 |
| | | 5 | 0.020686 | -0.085204 |
| | | 6 | 0.043677 | -0.065581 |
| | | 7 | 0.070830 | -0.039995 |
| | | 8 | 0.944372 | 0.623896 |
| E(LU) | | | 0.14885020 | |
| E(CP) | | | 0.04022462 | |
| E(LB) | | | | 0.10512398 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 13 | 9 | 1 | -0.017389 | -0.095590 |
| | | 2 | -0.008934 | -0.099109 |
| | | 3 | 0.001863 | -0.096521 |
| | | 4 | 0.014684 | -0.089554 |
| | | 5 | 0.029637 | -0.078490 |
| | | 6 | 0.047027 | -0.063068 |
| | | 7 | 0.067346 | -0.042607 |
| | | 8 | 0.091328 | -0.015928 |
| | | 9 | 0.774437 | 0.580865 |
| E(LU) | | | 0.12250342 | |
| E(CP) | | | 0.02046326 | |
| E(LB) | | | | 0.09030201 |
| 13 | 10 | 1 | -0.002927 | -0.083170 |
| | | 2 | 0.005067 | -0.087085 |
| | | 3 | 0.014356 | -0.085792 |
| | | 4 | 0.024891 | -0.080789 |
| | | 5 | 0.036816 | -0.072325 |
| | | 6 | 0.050389 | -0.060181 |
| | | 7 | 0.065995 | -0.043768 |
| | | 8 | 0.084201 | -0.022048 |
| | | 9 | 0.105863 | 0.006715 |
| | | 10 | 0.615348 | 0.528441 |
| E(LU) | | | 0.10607774 | |
| E(CP) | | | 0.00635741 | |
| E(LB) | | | | 0.07818835 |
| 13 | 11 | 1 | 0.007628 | -0.072617 |
| | | 2 | 0.015408 | -0.076746 |
| | | 3 | 0.023732 | -0.076418 |
| | | 4 | 0.032743 | -0.072938 |
| | | 5 | 0.042611 | -0.066531 |
| | | 6 | 0.053556 | -0.057014 |
| | | 7 | 0.065876 | -0.043886 |
| | | 8 | 0.080005 | -0.026244 |
| | | 9 | 0.096594 | -0.002552 |
| | | 10 | 0.116703 | 0.029910 |
| | | 11 | 0.465143 | 0.465037 |
| E(LU) | | | 0.09583611 | |
| E(CP) | | | -0.00388188 | |
| E(LB) | | | | 0.06795140 |
| 13 | 12 | 1 | 0.015382 | -0.063288 |
| | | 2 | 0.023100 | -0.067492 |
| | | 3 | 0.030818 | -0.067892 |
| | | 4 | 0.038824 | -0.065622 |
| | | 5 | 0.047302 | -0.060887 |
| | | 6 | 0.056444 | -0.053540 |
| | | 7 | 0.066482 | -0.043158 |
| | | 8 | 0.077739 | -0.028970 |
| | | 9 | 0.090699 | -0.009644 |
| | | 10 | 0.106166 | 0.017233 |
| | | 11 | 0.125627 | 0.056547 |
| | | 12 | 0.321416 | 0.386713 |
| E(LU) | | | 0.08961947 | |
| E(CP) | | | -0.01136145 | |
| E(LB) | | | | 0.05895232 |
| 13 | 13 | 1 | 0.021005 | -0.054436 |
| | | 2 | 0.028757 | -0.058585 |
| | | 3 | 0.036127 | -0.059535 |
| | | 4 | 0.043501 | -0.058259 |
| | | 5 | 0.051078 | -0.054942 |
| | | 6 | 0.059028 | -0.049472 |
| | | 7 | 0.067533 | -0.041504 |
| | | 8 | 0.076831 | -0.030398 |
| | | 9 | 0.087274 | -0.015037 |
| | | 10 | 0.099441 | 0.006644 |
| | | 11 | 0.114446 | 0.038943 |
| | | 12 | 0.135068 | 0.093324 |
| | | 13 | 0.179913 | 0.283257 |
| E(LU) | | | 0.08619744 | |
| E(CP) | | | -0.01674214 | |
| E(LB) | | | | 0.05046988 |
| 14 | 2 | 1 | -1.060461 | -0.490831 |
| | | 2 | 2.060461 | 0.490831 |
| E(LU) | | | 3.01527[illegible]98 | |
| E(CP) | | | 1.07852097 | |
| E(LB) | | | | 0.49075663 |
| 14 | 3 | 1 | -0.533185 | -0.324929 |
| | | 2 | -0.479874 | -0.308462 |
| | | 3 | 2.013059 | 0.633391 |
| E(LU) | | | 1.29865775 | |
| E(CP) | | | 0.53840104 | |
| E(LB) | | | | 0.32081269 |
| 14 | 4 | 1 | -0.310144 | -0.240992 |
| | | 2 | -0.281132 | -0.233670 |
| | | 3 | -0.229990 | -0.210735 |
| | | 4 | 1.821266 | 0.685397 |
| E(LU) | | | 0.69748231 | |
| E(CP) | | | 0.31216081 | |
| E(LB) | | | | 0.23567174 |
| 14 | 5 | 1 | -0.192947 | -0.190068 |
| | | 2 | -0.174709 | -0.187427 |
| | | 3 | -0.142478 | -0.172710 |
| | | 4 | -0.099930 | -0.149393 |
| | | 5 | 1.610065 | 0.699598 |
| E(LU) | | | 0.42609561 | |
| E(CP) | | | 0.19423903 | |
| E(LB) | | | | 0.18443288 |
| 14 | 6 | 1 | -0.123352 | -0.155736 |
| | | 2 | -0.110490 | -0.155747 |
| | | 3 | -0.088443 | -0.146054 |
| | | 4 | -0.059523 | -0.129460 |
| | | 5 | -0.024111 | -0.106556 |
| | | 6 | 1.405919 | 0.693553 |
| E(LU) | | | 0.28511973 | |
| E(CP) | | | 0.12469427 | |
| E(LB) | | | | 0.15012578 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | | C(N,M,I) |
|---|---|---|---|---|---|
| 14 | 7 | 1 | -0.078656 | | -0.130915 |
| | | 2 | -0.069666 | | -0.132521 |
| | | 3 | -0.052554 | | -0.126123 |
| | | 4 | -0.031776 | | -0.114051 |
| | | 5 | -0.006522 | | -0.096788 |
| | | 6 | 0.023467 | | -0.074184 |
| | | 7 | 1.214708 | | 0.674581 |
| E(LU) | | | 0.20518434 | | |
| E(CP) | | | | 0.08030259 | |
| E(LB) | | | | | 0.12547311 |
| 14 | 8 | 1 | -0.048365 | | -0.112041 |
| | | 2 | -0.039964 | | -0.114637 |
| | | 3 | -0.027495 | | -0.110509 |
| | | 4 | -0.011849 | | -0.101635 |
| | | 5 | 0.006905 | | -0.088422 |
| | | 6 | 0.029002 | | -0.070735 |
| | | 7 | 0.054897 | | -0.048074 |
| | | 8 | 1.036868 | | 0.646052 |
| E(LU) | | | 0.15716466 | | |
| E(CP) | | | | 0.05038249 | |
| E(LB) | | | | | 0.10683049 |
| 14 | 9 | 1 | -0.027030 | | -0.097117 |
| | | 2 | -0.019516 | | -0.100334 |
| | | 3 | -0.009363 | | -0.097827 |
| | | 4 | 0.002928 | | -0.091298 |
| | | 5 | 0.017368 | | -0.081103 |
| | | 6 | 0.034165 | | -0.067124 |
| | | 7 | 0.053685 | | -0.048921 |
| | | 8 | 0.076476 | | -0.025720 |
| | | 9 | 0.871267 | | 0.609445 |
| E(LU) | | | 0.12719148 | | |
| E(CP) | | | | 0.02941694 | |
| E(LB) | | | | | 0.09216556 |
| 14 | 10 | 1 | -0.011580 | | -0.084931 |
| | | 2 | -0.004548 | | -0.086528 |
| | | 3 | 0.004100 | | -0.087207 |
| | | 4 | 0.014144 | | -0.082451 |
| | | 5 | 0.025647 | | -0.074573 |
| | | 6 | 0.038794 | | -0.063473 |
| | | 7 | 0.053879 | | -0.048768 |
| | | 8 | 0.071335 | | -0.029776 |
| | | 9 | 0.091783 | | -0.005398 |
| | | 10 | 0.716445 | | 0.565105 |
| E(LU) | | | 0.10903536 | | |
| E(CP) | | | | 0.01430729 | |
| E(LB) | | | | | 0.08024763 |
| 14 | 11 | 1 | -0.000170 | | -0.074686 |
| | | 2 | 0.006622 | | -0.078499 |
| | | 3 | 0.014283 | | -0.078064 |
| | | 4 | 0.022800 | | -0.074680 |
| | | 5 | 0.032273 | | -0.068624 |
| | | 6 | 0.042866 | | -0.059816 |
| | | 7 | 0.054817 | | -0.047926 |
| | | 8 | 0.068463 | | -0.032355 |
| | | 9 | 0.084290 | | -0.012126 |
| | | 10 | 0.103025 | | 0.014349 |
| | | 11 | 0.570731 | | 0.512429 |
| E(LU) | | | 0.09566494 | | |
| E(CP) | | | | 0.00320055 | |
| E(LB) | | | | | 0.07027548 |
| 14 | 12 | 1 | 0.008361 | | -0.065816 |
| | | 2 | 0.015058 | | -0.069728 |
| | | 3 | 0.022076 | | -0.069962 |
| | | 4 | 0.029552 | | -0.067659 |
| | | 5 | 0.037615 | | -0.063070 |
| | | 6 | 0.046411 | | -0.056130 |
| | | 7 | 0.056132 | | -0.046558 |
| | | 8 | 0.067039 | | -0.033834 |
| | | 9 | 0.079506 | | -0.017101 |
| | | 10 | 0.094096 | | 0.005064 |
| | | 11 | 0.111723 | | 0.035156 |
| | | 12 | 0.432431 | | 0.449638 |
| E(LU) | | | 0.08771669 | | |
| E(CP) | | | | -0.00506397 | |
| E(LB) | | | | | 0.06168210 |
| 14 | 13 | 1 | 0.014760 | | -0.057849 |
| | | 2 | 0.021453 | | -0.061764 |
| | | 3 | 0.028064 | | -0.062506 |
| | | 4 | 0.034842 | | -0.061074 |
| | | 5 | 0.041933 | | -0.057693 |
| | | 6 | 0.049474 | | -0.052317 |
| | | 7 | 0.057619 | | -0.044707 |
| | | 8 | 0.066569 | | -0.034420 |
| | | 9 | 0.076605 | | -0.020713 |
| | | 10 | 0.088151 | | -0.002338 |
| | | 11 | 0.101914 | | 0.022943 |
| | | 12 | 0.119200 | | 0.059643 |
| | | 13 | 0.299416 | | 0.372795 |
| E(LU) | | | 0.08276211 | | |
| E(CP) | | | | -0.01123278 | |
| E(LB) | | | | | 0.05400148 |
| 14 | 14 | 1 | 0.019487 | | -0.050186 |
| | | 2 | 0.026238 | | -0.054008 |
| | | 3 | 0.032614 | | -0.055130 |
| | | 4 | 0.038947 | | -0.054419 |
| | | 5 | 0.045399 | | -0.052075 |
| | | 6 | 0.052097 | | -0.0480[illegible]6 |
| | | 7 | 0.059168 | | -0.042197 |
| | | 8 | 0.066767 | | -0.034099 |
| | | 9 | 0.075102 | | -0.023149 |
| | | 10 | 0.084482 | | -0.008285 |
| | | 11 | 0.095426 | | 0.012430 |
| | | 12 | 0.108942 | | 0.043015 |
| | | 13 | 0.127523 | | 0.094166 |
| | | 14 | 0.167807 | | 0.2720[illegible]4 |
| E(LU) | | | 0.07996685 | | |
| E(CP) | | | | -0.01576372 | |
| E(LB) | | | | | 0.04665712 |
| 15 | 2 | 1 | -1.097617 | | -0.491458 |
| | | 2 | 2.097617 | | 0.491458 |
| E(LU) | | | 3.17256460 | | |
| E(CP) | | | | 1.11445612 | |
| E(LB) | | | | | 0.49139327 |
| 15 | 3 | 1 | -0.55833[illegible] | | -0.325521 |
| | | 2 | -0.507671 | | -0.310191 |
| | | 3 | 2.066007 | | 0.635712 |
| E(LU) | | | 1.38015851 | | |
| E(CP) | | | | 0.56295169 | |
| E(LB) | | | | | 0.32168886 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 15 | 4 | 1 | -0.329324 | -0.241651 |
| | | 2 | -0.301829 | -0.234806 |
| | | 3 | -0.252948 | -0.213548 |
| | | 4 | 1.884101 | 0.690005 |

E(LU) 0.74642859
E(CP) 0.33084387
E(LB) 0.23669248

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 15 | 5 | 1 | -0.208525 | -0.190823 |
| | | 2 | -0.191357 | -0.188323 |
| | | 3 | -0.160491 | -0.174645 |
| | | 4 | -0.119743 | -0.153153 |
| | | 5 | 1.680121 | 0.706944 |

E(LU) 0.45754555
E(CP) 0.20933279
E(LB) 0.18556433

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 15 | 6 | 1 | -0.136498 | -0.156597 |
| | | 2 | -0.124518 | -0.156563 |
| | | 3 | -0.103401 | -0.147517 |
| | | | -0.075614 | -0.132162 |
| | | [illegible] | -0.041680 | -0.111215 |
| | | [illegible] | 1.481712 | 0.704074 |

E(LU) 0.30614004
E(CP) 0.13734100
E(LB) 0.15135556

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 15 | 7 | 1 | -0.090036 | -0.141891 |
| | | 2 | -0.080850 | -0.133342 |
| | | 3 | -0.065446 | -0.127335 |
| | | 4 | -0.045441 | -0.116138 |
| | | 5 | -0.021137 | -0.100291 |
| | | 6 | 0.007597 | -0.079774 |
| | | 7 | 1.295312 | 0.688771 |

E(LU) 0.21929214
E(CP) 0.09116039
E(LB) 0.12679942

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 15 | 8 | 1 | -0.058390 | -0.113143 |
| | | 2 | -0.050767 | -0.115520 |
| | | 3 | -0.038897 | -0.111607 |
| | | 4 | -0.023825 | -0.103332 |
| | | 5 | -0.005717 | -0.091156 |
| | | 6 | 0.015565 | -0.075053 |
| | | 7 | 0.040351 | -0.054703 |
| | | 8 | 1.121680 | 0.664514 |

E(LU) 0.16646559
E(CP) 0.05986446
E(LB) 0.10825884

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 15 | 9 | 1 | -0.035972 | -0.098361 |
| | | 2 | -0.029235 | -0.101322 |
| | | 3 | -0.019633 | -0.098904 |
| | | 4 | -0.007812 | -0.092773 |
| | | 5 | 0.006156 | -0.083327 |
| | | 6 | 0.022403 | -0.070544 |
| | | 7 | 0.041203 | -0.054142 |
| | | 8 | 0.062969 | -0.033595 |
| | | 9 | 0.959920 | 0.632967 |

E(LU) 0.13300106
E(CP) 0.03779810
E(LB) 0.09370837

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 15 | 10 | 1 | -0.019626 | -0.086339 |
| | | 2 | -0.013383 | -0.089664 |
| | | 3 | -0.005271 | -0.088341 |
| | | 4 | 0.004351 | -0.083828 |
| | | 5 | 0.015475 | -0.076474 |
| | | 6 | 0.028227 | -0.066261 |
| | | 7 | 0.042832 | -0.052943 |
| | | 8 | 0.059624 | -0.036054 |
| | | 9 | 0.079072 | -0.014863 |
| | | 10 | 0.808700 | 0.594768 |

E(LU) 0.11121862
E(CP) 0.02177795
E(LB) 0.08192616

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 15 | 11 | 1 | -0.007450 | -0.076297 |
| | | 2 | -0.001467 | -0.079835 |
| | | 3 | 0.005652 | -0.079332 |
| | | 4 | 0.013759 | -0.076068 |
| | | 5 | 0.022893 | -0.070355 |
| | | 6 | 0.033174 | -0.062181 |
| | | 7 | 0.044787 | -0.051331 |
| | | 8 | 0.057997 | -0.037396 |
| | | 9 | 0.073180 | -0.019723 |
| | | 10 | 0.090865 | 0.002701 |
| | | 11 | 0.666610 | 0.549817 |

E(LU) 0.09681113
E(CP) 0.00989471
E(LB) 0.07212492

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 15 | 12 | 1 | 0.001756 | -0.067695 |
| | | 2 | 0.007624 | -0.071342 |
| | | 3 | 0.014079 | -0.071459 |
| | | 4 | 0.021133 | -0.069178 |
| | | 5 | 0.028861 | -0.064779 |
| | | 6 | 0.037374 | -0.058256 |
| | | 7 | 0.046827 | -0.049425 |
| | | 8 | 0.057431 | -0.037926 |
| | | 9 | 0.069479 | -0.023180 |
| | | 10 | 0.083393 | -0.004[illegible] |
| | | 11 | 0.099799 | 0.[illegible] |
| | | 12 | 0.532243 | 0.497284 |

E(LU) 0.08723346
E(CP) 0.00094612
E(LB) 0.06376409

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 15 | 13 | 1 | 0.008779 | -0.060130 |
| | | 2 | 0.014620 | -0.063805 |
| | | 3 | 0.020637 | -0.064394 |
| | | [illegible] | 0.026961 | -0.062900 |
| | | 5 | 0.033693 | -0.059574 |
| | | 6 | 0.040939 | -0.054417 |
| | | 7 | 0.048828 | -0.047269 |
| | | 8 | 0.057528 | -0.037821 |
| | | 9 | 0.067265 | -0.025565 |
| | | 10 | 0.078368 | -0.009694 |
| | | 11 | 0.091330 | 0.011113 |
| | | 12 | 0.106947 | 0.039155 |
| | | 13 | 0.404106 | 0.435302 |

E(LU) 0.08092217
E(CP) -0.00585240
E(LB) 0.05644073

TABLE OF WEIGHTS (CONTINUED)

| | N | M | I | A(N,M,I) | | C(N,M,I) |
|---|---|---|---|---|---|---|
| | 15 | 14 | 1 | 0.014143 | | -0.053241 |
| | | | 2 | 0.020013 | | -0.056879 |
| | | | 3 | 0.025750 | | -0.057827 |
| | | | 4 | 0.031576 | | -0.056973 |
| | | | 5 | 0.037611 | | -0.054542 |
| | | | 6 | 0.043958 | | -0.050539 |
| | | | 7 | 0.050725 | | -0.044833 |
| | | | 8 | 0.058045 | | -0.037157 |
| | | | 9 | 0.066092 | | -0.027072 |
| | | | 10 | 0.075114 | | -0.013872 |
| | | | 11 | 0.085490 | | 0.003612 |
| | | | 12 | 0.097844 | | 0.027465 |
| | | | 13 | 0.113340 | | 0.061879 |
| | | | 14 | 0.280298 | | 0.359980 |
| E(LU) | | | | 0.07689745 | | |
| E(CP) | | | | | -0.01102126 | |
| E(LB) | | | | | | 0.04980248 |
| | 15 | 15 | 1 | 0.018170 | | -0.046538 |
| | | | 2 | 0.024108 | | -0.050064 |
| | | | 3 | 0.029685 | | -0.051279 |
| | | | 4 | 0.035191 | | -0.050957 |
| | | | 5 | 0.040762 | | -0.049293 |
| | | | 6 | 0.046496 | | -0.046315 |
| | | | 7 | 0.052488 | | -0.041899 |
| | | | 8 | 0.058844 | | -0.035827 |
| | | | 9 | 0.065696 | | -0.027731 |
| | | | 10 | 0.073230 | | -0.017008 |
| | | | 11 | 0.081725 | | -0.002653 |
| | | | 12 | 0.091651 | | 0.017156 |
| | | | 13 | 0.103914 | | 0.046191 |
| | | | 14 | 0.120784 | | 0.094483 |
| | | | 15 | 0.157255 | | 0.261738 |
| E(LU) | | | | 0.074[illegible]7775 | | |
| E(CP) | | | | | -0.01488220 | |
| E(LB) | | | | | | 0.04337628 |
| | 16 | 2 | 1 | -1.132243 | | -0.492005 |
| | | | 2 | 2.132243 | | 0.492005 |
| E(LU) | | | | 3.32404220 | | |
| E(CP) | | | | | 1.14801534 | |
| E(LB) | | | | | | 0.49194784 |
| | 16 | 3 | 1 | -0.581757 | | -0.326035 |
| | | | 2 | -0.533457 | | -0.311694 |
| | | | 3 | 2.115214 | | 0.637730 |
| E(LU) | | | | 1.45930438 | | |
| E(CP) | | | | | 0.58582769 | |
| E(LB) | | | | | | 0.32245028 |
| | 16 | 4 | 1 | -0.347172 | | -0.242220 |
| | | | 2 | -0.321026 | | -0.235794 |
| | | | 3 | -0.274186 | | -0.215984 |
| | | | 4 | 1.942384 | | 0.693998 |
| E(LU) | | | | 0.79453329 | | |
| E(CP) | | | | | 0.34828173 | |
| E(LB) | | | | | | 0.23757701 |
| | 16 | 5 | 1 | -0.223015 | | -0.191470 |
| | | | 2 | -0.206788 | | -0.189099 |
| | | | 3 | -0.177158 | | -0.176323 |
| | | | 4 | -0.138048 | | -0.156390 |
| | | | 5 | 1.745009 | | 0.713282 |
| E(LU) | | | | 0.48908000 | | |
| E(CP) | | | | | 0.22342597 | |
| E(LB) | | | | | | 0.18654151 |
| | 16 | 6 | 1 | -0.148725 | | -0.157331 |
| | | | 2 | -0.137508 | | -0.157263 |
| | | | 3 | -0.117232 | | -0.148785 |
| | | | 4 | -0.090481 | | -0.134532 |
| | | | 5 | -0.057883 | | -0.115196 |
| | | | 6 | 1.551828 | | 0.713108 |
| E(LU) | | | | 0.32746210 | | |
| E(CP) | | | | | 0.14915808 | |
| E(LB) | | | | | | 0.15241337 |
| | 16 | 7 | 1 | -0.100621 | | -0.132718 |
| | | | 2 | -0.092121 | | -0.134040 |
| | | | 3 | -0.077354 | | -0.128381 |
| | | | 4 | -0.058057 | | -0.117942 |
| | | | 5 | -0.034624 | | -0.103296 |
| | | | 6 | -0.007020 | | -0.084506 |
| | | | 7 | 1.369798 | | 0.700883 |
| E(LU) | | | | 0.23396225 | | |
| E(CP) | | | | | 0.10131710 | |
| E(LB) | | | | | | 0.12793461 |
| | 16 | 8 | 1 | -0.067719 | | -0.114069 |
| | | | 2 | -0.060754 | | -0.116260 |
| | | | 3 | -0.049415 | | -0.112545 |
| | | | 4 | 0.034868 | | -0.104798 |
| | | | 5 | -0.017357 | | -0.093508 |
| | | | 6 | 0.003178 | | -0.078726 |
| | | | 7 | 0.026[illegible]73 | | -0.060251 |
| | | | 8 | 1.199363 | | 0.680159 |
| E(LU) | | | | 0.17650200 | | |
| E(CP) | | | | | 0.06874770 | |
| E(LB) | | | | | | 0.10947376 |
| | 16 | 9 | 1 | -0.044303 | | -0.099396 |
| | | | 2 | -0.038218 | | -0.102138 |
| | | | 3 | -0.029094 | | -0.099811 |
| | | | 4 | -0.017697 | | -0.094037 |
| | | | 5 | -0.004166 | | -0.085242 |
| | | | 6 | 0.011570 | | -0.073467 |
| | | | 7 | 0.029712 | | -0.058535 |
| | | | 8 | 0.050576 | | -0.040084 |
| | | | 9 | 1.041619 | | 0.652711 |
| E(LU) | | | | 0.13966768 | | |
| E(CP) | | | | | 0.04566615 | |
| E(LB) | | | | | | 0.09501012 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 16 | 10 | 1 | -0.027135 | -0.087496 |
| | | 2 | -0.021550 | -0.090585 |
| | | 3 | -0.013895 | -0.089277 |
| | | 4 | -0.004646 | -0.084992 |
| | | 5 | 0.006132 | -0.078105 |
| | | 6 | 0.018515 | -0.068653 |
| | | 7 | 0.032675 | -0.056482 |
| | | 8 | 0.048869 | -0.041268 |
| | | 9 | 0.067459 | -0.022503 |
| | | 10 | 0.893576 | 0.619360 |
| E(LU) | | | 0.11534960 | |
| E(CP) | | | 0.02881067 | |
| E(LB) | | | | 0.08332716 |
| 16 | 11 | 1 | -0.014263 | -0.077597 |
| | | 2 | -0.008950 | -0.080895 |
| | | 3 | -0.002286 | -0.080349 |
| | | 4 | 0.005469 | -0.077213 |
| | | 5 | 0.014303 | -0.071820 |
| | | 6 | 0.024297 | -0.064207 |
| | | 7 | 0.035593 | -0.054237 |
| | | 8 | 0.040404 | -0.041625 |
| | | 9 | 0.063020 | -0.025917 |
| | | 10 | 0.079847 | -0.006432 |
| | | 11 | 0.754566 | 0.580293 |
| E(LU) | | | 0.09897866 | |
| E(CP) | | | 0.01622073 | |
| E(LB) | | | | 0.07364497 |
| 16 | 12 | 1 | -0.004450 | -0.069172 |
| | | 2 | 0.000732 | -0.072584 |
| | | 3 | 0.006721 | -0.072615 |
| | | 4 | 0.013424 | -0.070383 |
| | | 5 | 0.020868 | -0.066184 |
| | | 6 | 0.029134 | -0.060054 |
| | | 7 | 0.038344 | -0.051876 |
| | | 8 | 0.048668 | -0.041398 |
| | | 9 | 0.060342 | -0.028216 |
| | | 10 | 0.073692 | -0.011716 |
| | | 11 | 0.089173 | 0.009035 |
| | | 12 | 0.623351 | 0.535164 |
| E(LU) | | | 0.08784015 | |
| E(CP) | | | 0.00665801 | |
| E(LB) | | | | 0.06543511 |
| 16 | 13 | 1 | 0.003118 | -0.061843 |
| | | 2 | 0.008256 | -0.065297 |
| | | 3 | 0.013789 | -0.065770 |
| | | 4 | 0.019747 | -0.064259 |
| | | 5 | 0.026189 | -0.061031 |
| | | 6 | 0.033196 | -0.056120 |
| | | 7 | 0.040872 | -0.049427 |
| | | 8 | 0.049357 | -0.040731 |
| | | 9 | 0.058836 | -0.029675 |
| | | 10 | 0.069568 | -0.015710 |
| | | 11 | 0.081920 | 0.002010 |
| | | 12 | 0.096438 | 0.024833 |
| | | 13 | 0.498713 | 0.483018 |
| E(LU) | | | 0.08025299 | |
| E(CP) | | | -0.00069037 | |
| E(LB) | | | | 0.05831799 |
| 16 | 14 | 1 | 0.008992 | -0.055309 |
| | | 2 | 0.014141 | -0.058750 |
| | | 3 | 0.019370 | -0.059563 |
| | | 4 | 0.024804 | -0.058635 |
| | | 5 | 0.030525 | -0.056208 |
| | | 6 | 0.036615 | -0.052317 |
| | | 7 | 0.043164 | -0.046878 |
| | | 8 | 0.050284 | -0.039699 |
| | | 9 | 0.05[illegible]124 | -0.030467 |
| | | 10 | 0.066884 | -0.018695 |
| | | 11 | 0.076854 | -0.003625 |
| | | 12 | 0.088469 | 0.015969 |
| | | 13 | 0.102433 | 0.042224 |
| | | 14 | 0.379341 | 0.421953 |
| E(LU) | | | 0.07514429 | |
| E(CP) | | | -0.00637294 | |
| E(LB) | | | | 0.05199709 |
| 16 | 15 | 1 | 0.013547 | -0.049291 |
| | | 2 | 0.018743 | -0.052670 |
| | | 3 | 0.023778 | -0.053739 |
| | | 4 | 0.02884 | -0.053290 |
| | | 5 | 0.034060 | -0.051538 |
| | | 6 | 0.039489 | -0.048520 |
| | | 7 | 0.045218 | -0.044164 |
| | | 8 | 0.051338 | -0.038307 |
| | | 9 | 0.057965 | -0.030678 |
| | | 10 | 0.065253 | -0.020850 |
| | | 11 | 0.073425 | -0.008153 |
| | | 12 | 0.082818 | 0.008503 |
| | | 13 | 0.093994 | 0.031075 |
| | | 14 | 0.107995 | 0.063476 |
| | | 15 | 0.263522 | 0.348149 |
| E(LU) | | | 0.07182155 | |
| E(CP) | | | -0.01076262 | |
| E(LB) | | | | 0.04619787 |
| 16 | 16 | 1 | 0.017016 | -0.043375 |
| | | 2 | 0.022284 | -0.046633 |
| | | 3 | 0.027208 | -0.047890 |
| | | 4 | 0.032046 | -0.047839 |
| | | 5 | 0.036912 | -0.046675 |
| | | 6 | 0.041887 | -0.044432 |
| | | 7 | 0.047042 | -0.041053 |
| | | 8 | 0.052455 | -0.036402 |
| | | 9 | 0.058216 | -0.030249 |
| | | 10 | 0.064444 | -0.022230 |
| | | 11 | 0.071304 | -0.011772 |
| | | 12 | 0.079051 | 0.002079 |
| | | 13 | 0.088111 | 0.021044 |
| | | 14 | 0.099315 | 0.048675 |
| | | 15 | 0.114733 | 0.094419 |
| | | 16 | 0.147977 | 0.252333 |
| E(LU) | | | 0.06987019 | |
| E(CP) | | | -0.01409012 | |
| E(LB) | | | | 0.04052374 |
| 17 | 2 | 1 | -1.164659 | -0.492486 |
| | | 2 | 2.164659 | 0.492486 |
| E(LU) | | | 3.47015408 | |
| E(CP) | | | 1.17949167 | |
| E(LB) | | | | 0.49243526 |

| N | M | I | A(N,M,I) | | C(N,M,I) |
|---|---|---|---|---|---|
| 17 | 3 | 1 | -0.603668 | | -0.326486 |
| | | 2 | -0.557497 | | -0.313014 |
| | | 3 | 2.161166 | | 0.639500 |
| E(LU) | | | 1.53642388 | | |
| E(CP) | | | | 0.60729095 | |
| E(LB) | | | | | 0.32311812 |
| 17 | 4 | 1 | -0.363861 | | -0.242716 |
| | | 2 | -0.338922 | | -0.236662 |
| | | 3 | -0.293934 | | -0.218114 |
| | | 4 | 1.996717 | | 0.697492 |
| E(LU) | | | 0.84174810 | | |
| E(CP) | | | | 0.36462724 | |
| E(LB) | | | | | 0.23835098 |
| 17 | 5 | 1 | -0.236557 | | -0.192031 |
| | | 2 | -0.221164 | | -0.189778 |
| | | 3 | -0.192661 | | -0.177793 |
| | | 4 | -0.155037 | | -0.159206 |
| | | 5 | 1.805419 | | 0.718809 |
| E(LU) | | | 0.52028442 | | |
| E(CP) | | | | 0.23663986 | |
| E(LB) | | | | | 0.18739415 |
| 17 | 6 | 1 | -0.160149 | | -0.157965 |
| | | 2 | -0.149601 | | -0.157871 |
| | | 3 | -0.130090 | | -0.149896 |
| | | 4 | -0.104290 | | -0.136581 |
| | | 5 | -0.072907 | | -0.118639 |
| | | 6 | 1.617037 | | 0.720952 |
| E(LU) | | | 0.34893506 | | |
| E(CP) | | | | 0.16024410 | |
| E(LB) | | | | | 0.15333326 |
| 17 | 7 | 1 | -0.110512 | | -0.133428 |
| | | 2 | -0.102606 | | -0.134640 |
| | | 3 | -0.088415 | | -0.129294 |
| | | 4 | -0.069771 | | -0.119517 |
| | | 5 | -0.047139 | | -0.105901 |
| | | 6 | -0.020560 | | -0.088568 |
| | | 7 | 1.439003 | | 0.711349 |
| E(LU) | | | 0.24902198 | | |
| E(CP) | | | | 0.11085361 | |
| E(LB) | | | | | 0.12891783 |
| 17 | 8 | 1 | -0.076441 | | -0.114859 |
| | | 2 | -0.070039 | | -0.116891 |
| | | 3 | -0.059173 | | -0.113357 |
| | | 4 | -0.045110 | | -0.106076 |
| | | 5 | -0.028154 | | -0.095554 |
| | | 6 | -0.008307 | | -0.081890 |
| | | 7 | 0.014595 | | -0.064968 |
| | | 8 | 1.272528 | | 0.693595 |
| E(LU) | | | 0.18708688 | | |
| E(CP) | | | | 0.07709833 | |
| E(LB) | | | | | 0.11052085 |
| 17 | 9 | 1 | -0.052096 | | -0.100271 |
| | | 2 | -0.046565 | | -0.102825 |
| | | 3 | -0.037862 | | -0.100587 |
| | | 4 | -0.026851 | | -0.095136 |
| | | 5 | -0.013728 | | -0.086910 |
| | | 6 | 0.001531 | | -0.075995 |
| | | 7 | 0.019069 | | -0.062288 |
| | | 8 | 0.039129 | | -0.045535 |
| | | 9 | 1.117373 | | 0.669546 |
| E(LU) | | | 0.14699387 | | |
| E(CP) | | | | 0.05307401 | |
| E(LB) | | | | | 0.09612512 |
| 17 | 10 | 1 | -0.034167 | | -0.088465 |
| | | 2 | -0.029139 | | -0.091350 |
| | | 3 | -0.021881 | | -0.090064 |
| | | 4 | -0.012965 | | -0.085992 |
| | | 5 | -0.002507 | | -0.079521 |
| | | 6 | 0.009531 | | -0.070728 |
| | | 7 | 0.023273 | | -0.059520 |
| | | 8 | 0.038922 | | -0.045671 |
| | | 9 | 0.056761 | | -0.028822 |
| | | 10 | 0.972172 | | 0.640135 |
| E(LU) | | | 0.12022174 | | |
| E(CP) | | | | 0.03544569 | |
| E(LB) | | | | | 0.08451762 |
| 17 | 11 | 1 | -0.020654 | | -0.078673 |
| | | 2 | -0.015906 | | -0.081761 |
| | | 3 | -0.009632 | | -0.081188 |
| | | 4 | -0.002186 | | -0.078180 |
| | | 5 | 0.006378 | | -0.073083 |
| | | 6 | 0.016104 | | -0.065964 |
| | | 7 | 0.027102 | | -0.056744 |
| | | 8 | 0.039540 | | -0.045224 |
| | | 9 | 0.053648 | | -0.031078 |
| | | 10 | 0.069744 | | -0.013827 |
| | | 11 | 0.835861 | | 0.605723 |
| E(LU) | | | 0.10195092 | | |
| E(CP) | | | | 0.02220540 | |
| E(LB) | | | | | 0.07492279 |
| 17 | 12 | 1 | -0.010286 | | -0.070375 |
| | | 2 | -0.005683 | | -0.073577 |
| | | 3 | -0.000086 | | -0.073546 |
| | | 4 | 0.006316 | | -0.071375 |
| | | 5 | 0.013511 | | -0.067372 |
| | | 6 | 0.021553 | | -0.061602 |
| | | 7 | 0.030535 | | -0.053996 |
| | | 8 | 0.040597 | | -0.044377 |
| | | 9 | 0.051928 | | -0.032455 |
| | | 10 | 0.064785 | | -0.017797 |
| | | 11 | 0.079517 | | 0.000228 |
| | | 12 | 0.707314 | | 0.566244 |
| E(LU) | | | 0.08930564 | | |
| E(CP) | | | | 0.01208216 | |
| E(LB) | | | | | 0.06681858 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 17 | 13 | 1 | -0.002231 | -0.063202 |
| | | 2 | 0.002318 | -0.066454 |
| | | 3 | 0.007448 | -0.066839 |
| | | 4 | 0.013101 | -0.065335 |
| | | 5 | 0.019298 | -0.062220 |
| | | 6 | 0.026098 | -0.057556 |
| | | 7 | 0.033584 | -0.051282 |
| | | 8 | 0.041872 | -0.043242 |
| | | 9 | 0.051113 | -0.033181 |
| | | 10 | 0.061516 | -0.020708 |
| | | 11 | 0.073364 | -0.005250 |
| | | 12 | 0.087058 | 0.014056 |
| | | 13 | 0.585461 | 0.521211 |
| E(LU) | | | 0.08049558 | |
| E(CP) | | | 0.00423893 | |
| E(LB) | | | | 0.05983608 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 17 | 14 | 1 | 0.004088 | -0.056878 |
| | | 2 | 0.008636 | -0.060131 |
| | | 3 | 0.013446 | -0.060836 |
| | | 4 | 0.018560 | -0.059871 |
| | | 5 | 0.024028 | -0.057487 |
| | | 6 | 0.029909 | -0.053742 |
| | | 7 | 0.036278 | -0.048586 |
| | | 8 | 0.043231 | -0.041881 |
| | | 9 | 0.050892 | -0.033402 |
| | | 10 | 0.059426 | -0.022709 |
| | | 11 | 0.069060 | -0.009558 |
| | | 12 | 0.080119 | 0.007[illegible]2 |
| | | 13 | 0.093083 | 0.028459 |
| | | 14 | 0.469244 | 0.569601 |
| E(LU) | | | 0.07436842 | |
| E(CP) | | | -0.00189289 | |
| E(LB) | | | | 0.05369960 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 17 | 15 | 1 | 0.009066 | -0.051176 |
| | | 2 | 0.013648 | -0.054390 |
| | | 3 | 0.018244 | -0.055341 |
| | | 4 | 0.022974 | -0.054815 |
| | | 5 | 0.027908 | -0.053042 |
| | | 6 | 0.033111 | -0.050075 |
| | | 7 | 0.038648 | -0.045871 |
| | | 8 | 0.044600 | -0.040314 |
| | | 9 | 0.051065 | -0.033203 |
| | | 10 | 0.058176 | -0.024231 |
| | | 11 | 0.066111 | -0.012936 |
| | | 12 | 0.075128 | 0.001394 |
| | | 13 | 0.085616 | 0.019905 |
| | | 14 | 0.098200 | 0.044587 |
| | | 15 | 0.357506 | 0.409507 |
| E(LU) | | | 0.07016498 | |
| E(CP) | | | -0.00670775 | |
| E(LB) | | | | 0.04818440 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 17 | 16 | 1 | 0.012979 | -0.045870 |
| | | 2 | 0.017617 | -0.049009 |
| | | 3 | 0.022076 | -0.050145 |
| | | 4 | 0.026538 | -0.049982 |
| | | 5 | 0.031091 | -0.048727 |
| | | 6 | 0.035799 | -0.046430 |
| | | 7 | 0.040724 | -0.043057 |
| | | 8 | 0.045932 | -0.038508 |
| | | 9 | 0.051504 | -0.032609 |
| | | 10 | 0.057542 | -0.025090 |
| | | 11 | 0.064186 | -0.015545 |
| | | 12 | 0.071635 | -0.003341 |
| | | 13 | 0.080193 | 0.012556 |
| | | 14 | 0.090373 | 0.033974 |
| | | 15 | 0.103110 | 0.064588 |
| | | 16 | 0.248699 | 0.337194 |
| E(LU) | | | 0.06738336 | |
| E(CP) | | | -0.01047916 | |
| E(LB) | | | | 0.04307100 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 17 | 17 | 1 | 0.015998 | -0.040607 |
| | | 2 | 0.020706 | -0.043624 |
| | | 3 | 0.025089 | -0.044891 |
| | | 4 | 0.029378 | -0.045031 |
| | | 5 | 0.033671 | -0.044229 |
| | | 6 | 0.038035 | -0.042531 |
| | | 7 | 0.042527 | -0.039913 |
| | | 8 | 0.047204 | -0.036289 |
| | | 9 | 0.052133 | -0.031512 |
| | | 10 | 0.057392 | -0.025352 |
| | | 11 | 0.063089 | -0.017458 |
| | | 12 | 0.069375 | -0.007282 |
| | | 13 | 0.076482 | 0.006082 |
| | | 14 | 0.084803 | 0.024262 |
| | | 15 | 0.095098 | 0.050618 |
| | | 16 | 0.109270 | 0.094076 |
| | | 17 | 0.139752 | 0.243681 |
| E(LU) | | | 0.06572241 | |
| E(CP) | | | -0.01337530 | |
| E(LB) | | | | 0.03802109 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 18 | 2 | 1 | -1.195128 | -0.492912 |
| | | 2 | 2.195128 | 0.492912 |
| E(LU) | | | 3.61129585 | |
| E(CP) | | | 1.20912723 | |
| E(LB) | | | | 0.49286703 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 18 | 3 | 1 | -0.624252 | -0.326884 |
| | | 2 | -0.580008 | -0.314183 |
| | | 3 | 2.204260 | 0.641066 |
| E(LU) | | | 1.61137253 | |
| E(CP) | | | 0.62748837 | |
| E(LB) | | | | 0.32370865 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 18 | 4 | 1 | -0.379529 | -0.243153 |
| | | 2 | -0.355679 | -0.237429 |
| | | 3 | -0.312382 | -0.219992 |
| | | 4 | 2.047590 | 0.700574 |
| E(LU) | | | 0.88805128 | |
| E(CP) | | | 0.38000703 | |
| E(LB) | | | | 0.23903395 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 18 | 5 | 1 | -0.249266 | -0.192523 |
| | | 2 | -0.234618 | -0.190376 |
| | | 3 | -0.207148 | -0.179091 |
| | | 4 | -0.170883 | -0.161679 |
| | | 5 | 1.861914 | 0.723670 |
| E(LU) | | | 0.55118001 | |
| E(CP) | | | 0.24907530 | |
| E(LB) | | | | 0.18814472 |
| 18 | 6 | 1 | -0.170868 | -0.158518 |
| | | 2 | -0.160910 | -0.158405 |
| | | 3 | -0.142100 | -0.150876 |
| | | 4 | -0.117175 | -0.138383 |
| | | 5 | -0.086906 | -0.121647 |
| | | 6 | 1.677960 | 0.727829 |
| E(LU) | | | 0.37044855 | |
| E(CP) | | | 0.17068152 | |
| E(LB) | | | | 0.15414076 |
| 18 | 7 | 1 | -0.119793 | -0.134044 |
| | | 2 | -0.112406 | -0.135163 |
| | | 3 | -0.098738 | -0.130098 |
| | | 4 | -0.080698 | -0.120904 |
| | | 5 | -0.058807 | -0.108183 |
| | | 6 | -0.033165 | -0.092095 |
| | | 7 | 1.503605 | 0.720486 |
| E(LU) | | | 0.26434202 | |
| E(CP) | | | 0.11983820 | |
| E(LB) | | | | 0.12977806 |
| 18 | 8 | 1 | -0.084626 | -0.115541 |
| | | 2 | -0.078711 | -0.117434 |
| | | 3 | -0.068272 | -0.114068 |
| | | 4 | -0.054656 | -0.107202 |
| | | 5 | -0.038217 | -0.097349 |
| | | 6 | -0.019006 | -0.084645 |
| | | 7 | 0.003084 | -0.069031 |
| | | 8 | 1.340405 | 0.705270 |
| E(LU) | | | 0.19807869 | |
| E(CP) | | | 0.08497296 | |
| E(LB) | | | | 0.11143330 |
| 18 | 9 | 1 | -0.059414 | -0.101022 |
| | | 2 | -0.054359 | -0.103411 |
| | | 3 | -0.046030 | -0.101260 |
| | | 4 | -0.035375 | -0.096099 |
| | | 5 | -0.022631 | -0.088374 |
| | | 6 | -0.007819 | -0.078203 |
| | | 7 | 0.009161 | -0.065532 |
| | | 8 | 0.028495 | -0.050186 |
| | | 9 | 1.187973 | 0.684067 |
| E(LU) | | | 0.15482946 | |
| E(CP) | | | 0.06006815 | |
| E(LB) | | | | 0.09709201 |
| 18 | 10 | 1 | -0.040776 | -0.089291 |
| | | 2 | -0.036223 | -0.091997 |
| | | 3 | -0.029314 | -0.090739 |
| | | 4 | -0.020701 | -0.086863 |
| | | 5 | -0.010540 | -0.080764 |
| | | 6 | 0.001172 | -0.072544 |
| | | 7 | 0.014523 | -0.062157 |
| | | 8 | 0.029671 | -0.049465 |
| | | 9 | 0.046841 | -0.034147 |
| | | 10 | 1.045347 | 0.657947 |
| E(LU) | | | 0.12567798 | |
| E(CP) | | | 0.04172006 | |
| E(LB) | | | | 0.08554362 |
| 18 | 11 | 1 | -0.026669 | -0.079582 |
| | | 2 | -0.022402 | -0.082484 |
| | | 3 | -0.016466 | -0.081896 |
| | | 4 | -0.009294 | -0.079012 |
| | | 5 | -0.000979 | -0.074183 |
| | | 6 | 0.008496 | -0.067503 |
| | | 7 | 0.019212 | -0.058930 |
| | | 8 | 0.031300 | -0.048324 |
| | | 9 | 0.044947 | -0.035491 |
| | | 10 | 0.060404 | -0.019962 |
| | | 11 | 0.911469 | 0.627325 |
| E(LU) | | | 0.10556433 | |
| E(CP) | | | 0.02787638 | |
| E(LB) | | | | 0.07601539 |
| 18 | 12 | 1 | -0.015793 | -0.071378 |
| | | 2 | -0.011677 | -0.074393 |
| | | 3 | -0.006416 | -0.074315 |
| | | 4 | -0.000278 | -0.072211 |
| | | 5 | 0.006695 | -0.068395 |
| | | 6 | 0.014529 | -0.062952 |
| | | 7 | 0.023297 | -0.055848 |
| | | 8 | 0.033110 | -0.046959 |
| | | 9 | 0.044122 | -0.036073 |
| | | 10 | 0.056540 | -0.022877 |
| | | 11 | 0.070637 | -0.006924 |
| | | 12 | 0.785235 | 0.592326 |
| E(LU) | | | 0.09145851 | |
| E(CP) | | | 0.01723593 | |
| E(LB) | | | | 0.06798899 |
| 18 | 13 | 1 | -0.007289 | -0.064317 |
| | | 2 | -0.003236 | -0.067387 |
| | | 3 | 0.001550 | -0.067701 |
| | | 4 | 0.006940 | -0.066218 |
| | | 5 | 0.012925 | -0.063222 |
| | | 6 | 0.019540 | -0.058792 |
| | | 7 | 0.026851 | -0.052898 |
| | | 8 | 0.034951 | -0.045430 |
| | | 9 | 0.043969 | -0.036200 |
| | | 10 | 0.054072 | -0.024926 |
| | | 11 | 0.065486 | -0.011201 |
| | | 12 | 0.070516 | 0.005561 |
| | | 13 | 0.665728 | 0.552731 |
| E(LU) | | | 0.08146655 | |
| E(CP) | | | 0.00893995 | |
| E(LB) | | | | 0.06110111 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 18 | 14 | 1 | -0.000568 | -0.058133 |
| | | 2 | 0.003471 | -0.061213 |
| | | 3 | 0.007930 | -0.061830 |
| | | 4 | 0.012775 | -0.060849 |
| | | 5 | 0.018027 | -0.058527 |
| | | 6 | 0.023730 | -0.054936 |
| | | 7 | 0.029942 | -0.050053 |
| | | 8 | 0.036744 | -0.043781 |
| | | 9 | 0.044239 | -0.035952 |
| | | 10 | 0.052564 | -0.026314 |
| | | 11 | 0.061904 | -0.014497 |
| | | 12 | 0.072509 | 0.003034 |
| | | 13 | 0.084730 | 0.018080 |
| | | 14 | 0.552004 | 0.507970 |
| E(LU) | | | 0.07436294 | |
| E(CP) | | | 0.00240300 | |
| E(LB) | | | | 0.05508562 |
| 18 | 15 | 1 | 0.004780 | -0.052617 |
| | | 2 | 0.008843 | -0.055674 |
| | | 3 | 0.013074 | -0.056526 |
| | | 4 | 0.017522 | -0.055953 |
| | | 5 | 0.022232 | -0.054191 |
| | | 6 | 0.027249 | -0.051307 |
| | | 7 | 0.032630 | -0.047281 |
| | | 8 | 0.038443 | -0.042029 |
| | | 9 | 0.044772 | -0.035403 |
| | | 10 | 0.051728 | -0.027176 |
| | | 11 | 0.059460 | -0.017018 |
| | | 12 | 0.068169 | -0.004442 |
| | | 13 | 0.078145 | 0.011289 |
| | | 14 | 0.089813 | 0.031340 |
| | | 15 | 0.443142 | 0.456986 |
| E(LU) | | | 0.06933298 | |
| E(CP) | | | -0.00278409 | |
| E(LB) | | | | 0.04973648 |
| 18 | 16 | 1 | 0.009048 | -0.047594 |
| | | 2 | 0.013157 | -0.050597 |
| | | 3 | 0.017235 | -0.051629 |
| | | 4 | 0.021397 | -0.051393 |
| | | 5 | 0.025706 | -0.050102 |
| | | 6 | 0.030212 | -0.047820 |
| | | 7 | 0.034966 | -0.044532 |
| | | 8 | 0.040027 | -0.040165 |
| | | 9 | 0.045465 | -0.034587 |
| | | 10 | 0.051368 | -0.027599 |
| | | 11 | 0.057855 | -0.018906 |
| | | 12 | 0.065087 | -0.008069 |
| | | 13 | 0.073294 | 0.005581 |
| | | 14 | 0.082827 | 0.023119 |
| | | 15 | 0.094248 | 0.046408 |
| | | 16 | 0.338109 | 0.397887 |
| E(LU) | | | 0.06582537 | |
| E(CP) | | | -0.00691185 | |
| E(LB) | | | | 0.04487895 |
| 18 | 17 | 1 | 0.012444 | -0.042879 |
| | | 2 | 0.016611 | -0.045800 |
| | | 3 | 0.020593 | -0.046965 |
| | | 4 | 0.024555 | -0.047008 |
| | | 5 | 0.028573 | -0.046121 |
| | | 6 | 0.032702 | -0.044362 |
| | | 7 | 0.036990 | -0.041722 |
| | | 8 | 0.041487 | -0.038137 |
| | | 9 | 0.046252 | -0.033494 |
| | | 10 | 0.051355 | -0.027618 |
| | | 11 | 0.056889 | -0.020248 |
| | | 12 | 0.062980 | -0.010994 |
| | | 13 | 0.069810 | 0.000742 |
| | | 14 | 0.077655 | 0.015937 |
| | | 15 | 0.086979 | 0.036313 |
| | | 16 | 0.098636 | 0.065331 |
| | | 17 | 0.235490 | 0.327023 |
| E(LU) | | | 0.06346845 | |
| E(CP) | | | -0.01018489 | |
| E(LB) | | | | 0.04033369 |
| 18 | 18 | 1 | 0.015092 | -0.038165 |
| | | 2 | 0.019328 | -0.040965 |
| | | 3 | 0.023258 | -0.042221 |
| | | 4 | 0.027089 | -0.042497 |
| | | 5 | 0.030909 | -0.041963 |
| | | 6 | 0.034773 | -0.040676 |
| | | 7 | 0.038728 | -0.038627 |
| | | 8 | 0.042820 | -0.035765 |
| | | 9 | 0.047095 | -0.031992 |
| | | 10 | 0.051612 | -0.027160 |
| | | 11 | 0.056443 | -0.021041 |
| | | 12 | 0.061685 | -0.013300 |
| | | 13 | 0.067477 | -0.003410 |
| | | 14 | 0.074032 | 0.009488 |
| | | 15 | 0.081713 | 0.026940 |
| | | 16 | 0.091221 | 0.052132 |
| | | 17 | 0.104314 | 0.093529 |
| | | 18 | 0.132411 | 0.235693 |
| E(LU) | | | 0.06204005 | |
| E(CP) | | | -0.01272745 | |
| E(LB) | | | | 0.03580789 |
| 19 | 2 | 1 | -1.223869 | -0.493292 |
| | | 2 | 2.223869 | 0.493292 |
| E(LU) | | | 3.74782267 | |
| E(CP) | | | 1.23712437 | |
| E(LB) | | | | 0.49325215 |
| 19 | 3 | 1 | -0.643659 | -0.327238 |
| | | 2 | -0.601169 | -0.315224 |
| | | 3 | 2.244827 | 0.642462 |
| E(LU) | | | 1.68432765 | |
| E(CP) | | | 0.64655945 | |
| E(LB) | | | | 0.32423458 |
| 19 | 4 | 1 | -0.394294 | -0.243540 |
| | | 2 | -0.371431 | -0.238113 |
| | | 3 | -0.329685 | -0.221662 |
| | | 4 | 2.095409 | 0.703314 |
| E(LU) | | | 0.93343886 | |
| E(CP) | | | 0.39452713 | |
| E(LB) | | | | 0.23964110 |

| | N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|---|
| | 19 | 5 | 1 | -0.261237 | -0.192958 |
| | | | 2 | -0.247259 | -0.190909 |
| | | | 3 | -0.220739 | -0.180245 |
| | | | 4 | -0.185724 | -0.163869 |
| | | | 5 | 1.914959 | 0.727980 |
| E(LU) | | | | 0.58171310 | |
| E(CP) | | | | 0.26081700 | |
| E(LB) | | | | | 0.18861359 |
| | 19 | 6 | 1 | -0.180964 | -0.159004 |
| | | | 2 | -0.171530 | -0.158877 |
| | | | 3 | -0.153365 | -0.151748 |
| | | | 4 | -0.129250 | -0.139981 |
| | | | 5 | -0.100003 | -0.124297 |
| | | | 6 | 1.735111 | 0.733908 |
| E(LU) | | | | 0.39192137 | |
| E(CP) | | | | 0.18053993 | |
| E(LB) | | | | | 0.15435543 |
| | 19 | 7 | 1 | -0.128533 | -0.134583 |
| | | | 2 | -0.121602 | -0.135622 |
| | | | 3 | -0.108414 | -0.130811 |
| | | | 4 | -0.090935 | -0.122136 |
| | | | 5 | -0.069730 | -0.110197 |
| | | | 6 | -0.044949 | -0.095185 |
| | | | 7 | 1.564162 | 0.728535 |
| E(LU) | | | | 0.27982455 | |
| E(CP) | | | | 0.12832890 | |
| E(LB) | | | | | 0.13053726 |
| | 19 | 8 | 1 | -0.092336 | -0.116135 |
| | | | 2 | -0.086846 | -0.117908 |
| | | | 3 | -0.076795 | -0.114696 |
| | | | 4 | -0.063593 | -0.108201 |
| | | | 5 | -0.047637 | -0.098937 |
| | | | 6 | -0.029016 | -0.087065 |
| | | | 7 | -0.007670 | -0.072570 |
| | | | 8 | 1.403893 | 0.715513 |
| E(LU) | | | | 0.20936888 | |
| E(CP) | | | | 0.09242024 | |
| E(LB) | | | | | 0.11223593 |
| | 19 | 9 | 1 | -0.066309 | -0.101674 |
| | | | 2 | -0.061667 | -0.103918 |
| | | | 3 | -0.053675 | -0.101850 |
| | | | 4 | -0.043547 | -0.096952 |
| | | | 5 | -0.030960 | -0.089671 |
| | | | 6 | -0.016565 | -0.080147 |
| | | | 7 | -0.000103 | -0.068365 |
| | | | 8 | 0.018570 | -0.054204 |
| | | | 9 | 1.254056 | 0.696782 |
| E(LU) | | | | 0.16305851 | |
| E(CP) | | | | 0.06668914 | |
| E(LB) | | | | | 0.09793914 |
| | 19 | 10 | 1 | -0.047007 | -0.090004 |
| | | | 2 | -0.042865 | -0.092551 |
| | | | 3 | -0.036265 | -0.091324 |
| | | | 4 | -0.027929 | -0.087629 |
| | | | 5 | -0.018045 | -0.081863 |
| | | | 6 | -0.006641 | -0.074147 |
| | | | 7 | 0.006343 | -0.064468 |
| | | | 8 | 0.021029 | -0.052718 |
| | | | 9 | 0.037595 | -0.038703 |
| | | | 10 | 1.112786 | 0.673407 |
| E(LU) | | | | 0.13159684 | |
| E(CP) | | | | 0.04766707 | |
| E(LB) | | | | | 0.08643819 |
| | 19 | 11 | 1 | -0.032345 | -0.080360 |
| | | | 2 | -0.028492 | -0.083097 |
| | | | 3 | -0.022852 | -0.082502 |
| | | | 4 | -0.015928 | -0.079736 |
| | | | 5 | -0.007843 | -0.075152 |
| | | | 6 | 0.001395 | -0.068861 |
| | | | 7 | 0.011842 | -0.060851 |
| | | | 8 | 0.023603 | -0.051025 |
| | | | 9 | 0.036827 | -0.039209 |
| | | | 10 | 0.051716 | -0.025147 |
| | | | 11 | 0.982076 | 0.645940 |
| E(LU) | | | | 0.10969257 | |
| E(CP) | | | | 0.03326000 | |
| E(LB) | | | | | 0.07696225 |
| | 19 | 12 | 1 | -0.020995 | -0.072230 |
| | | | 2 | -0.017298 | -0.075078 |
| | | | 3 | -0.012331 | -0.074965 |
| | | | 4 | -0.006425 | -0.072929 |
| | | | 5 | 0.000344 | -0.069288 |
| | | | 6 | 0.007984 | -0.064141 |
| | | | 7 | 0.016548 | -0.057480 |
| | | | 8 | 0.026124 | -0.049218 |
| | | | 9 | 0.036839 | -0.039200 |
| | | | 10 | 0.048859 | -0.027194 |
| | | | 11 | 0.062404 | -0.012872 |
| | | | 12 | 0.857947 | 0.614595 |
| E(LU) | | | | 0.09416748 | |
| E(CP) | | | | 0.02213853 | |
| E(LB) | | | | | 0.06899532 |
| | 19 | 13 | 1 | -0.012077 | -0.065253 |
| | | | 2 | -0.008453 | -0.068158 |
| | | | 3 | -0.003960 | -0.068416 |
| | | | 4 | 0.001201 | -0.066962 |
| | | | 5 | 0.006996 | -0.064084 |
| | | | 6 | 0.013442 | -0.059872 |
| | | | 7 | 0.020588 | -0.054320 |
| | | | 8 | 0.028511 | -0.047351 |
| | | | 9 | 0.037316 | -0.038827 |
| | | | 10 | 0.047141 | -0.028538 |
| | | | 11 | 0.058169 | -0.016185 |
| | | | 12 | 0.070640 | -0.001353 |
| | | | 13 | 0.740487 | 0.579318 |
| E(LU) | | | | 0.08302831 | |
| E(CP) | | | | 0.01342381 | |
| E(LB) | | | | | 0.06217738 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 19 | 14 | 1 | -0.004989 | -0.059169 |
| | | 2 | -0.001384 | -0.062091 |
| | | 3 | 0.002773 | -0.062636 |
| | | 4 | 0.007387 | -0.061652 |
| | | 5 | 0.012453 | -0.059399 |
| | | 6 | 0.017997 | -0.055961 |
| | | 7 | 0.024066 | -0.051334 |
| | | 8 | 0.030726 | -0.045450 |
| | | 9 | 0.038064 | -0.038185 |
| | | 10 | 0.046194 | -0.029351 |
| | | 11 | 0.055267 | -0.018676 |
| | | 12 | 0.065482 | -0.005781 |
| | | 13 | 0.077109 | 0.009882 |
| | | 14 | 0.628854 | 0.539802 |
| E(LU) | | | 0.07498023 | |
| E(CP) | | | 0.00651543 | |
| E(LB) | | | | 0.05624730 |
| 19 | 15 | 1 | 0.000692 | -0.053779 |
| | | 2 | 0.004313 | -0.056686 |
| | | 3 | 0.008234 | -0.057456 |
| | | 4 | 0.012444 | -0.056855 |
| | | 5 | 0.016963 | -0.055121 |
| | | 6 | 0.021823 | -0.052332 |
| | | 7 | 0.027068 | -0.048486 |
| | | 8 | 0.032757 | -0.043523 |
| | | 9 | 0.038961 | -0.037334 |
| | | 10 | 0.045774 | -0.029749 |
| | | 11 | 0.053320 | -0.020523 |
| | | 12 | 0.061762 | -0.009310 |
| | | 13 | 0.071323 | 0.004394 |
| | | 14 | 0.082316 | 0.021334 |
| | | 15 | 0.522250 | 0.495426 |
| E(LU) | | | 0.06915784 | |
| E(CP) | | | 0.00099208 | |
| E(LB) | | | | 0.05100764 |
| 19 | 16 | 1 | 0.005271 | -0.043924 |
| | | 2 | 0.008929 | -0.051792 |
| | | 3 | 0.012687 | -0.052735 |
| | | 4 | 0.016601 | -0.052449 |
| | | 5 | 0.020708 | -0.051151 |
| | | 6 | 0.025048 | -0.048913 |
| | | 7 | 0.029663 | -0.045735 |
| | | 8 | 0.034603 | -0.041566 |
| | | 9 | 0.039929 | -0.036308 |
| | | 10 | 0.045717 | -0.029809 |
| | | 11 | 0.052068 | -0.0218[illegible]0 |
| | | 12 | 0.059114 | -0.012148 |
| | | 13 | 0.067036 | -0.000151 |
| | | 14 | 0.076094 | 0.014738 |
| | | 15 | 0.086669 | 0.033642 |
| | | 16 | 0.419861 | 0.445119 |
| E(LU) | | | 0.06496979 | |
| E(CP) | | | -0.00344791 | |
| E(LB) | | | | 0.04630055 |
| 19 | 17 | 1 | 0.008968 | -0.044464 |
| | | 2 | 0.012677 | -0.047270 |
| | | 3 | 0.016326 | -0.048344 |
| | | 4 | 0.020023 | -0.048319 |
| | | 5 | 0.023825 | -0.047390 |
| | | 6 | 0.027772 | -0.045626 |
| | | 7 | 0.031706 | -0.043028 |
| | | 8 | 0.036272 | -0.039552 |
| | | 9 | 0.040920 | -0.035112 |
| | | 10 | 0.045913 | -0.029573 |
| | | 11 | 0.051331 | -0.022740 |
| | | 12 | 0.057280 | -0.014330 |
| | | 13 | 0.063906 | -0.003928 |
| | | 14 | 0.071419 | 0.009098 |
| | | 15 | 0.080136 | 0.025759 |
| | | 16 | 0.090563 | 0.047806 |
| | | 17 | 0.320763 | 0.387016 |
| E(LU) | | | 0.06200679 | |
| E(CP) | | | -0.00702291 | |
| E(LB) | | | | 0.04198714 |
| 19 | 18 | 1 | 0.011941 | -0.040244 |
| | | 2 | 0.015709 | -0.042966 |
| | | 3 | 0.019289 | -0.044136 |
| | | 4 | 0.022835 | -0.044327 |
| | | 5 | 0.026412 | -0.043716 |
| | | 6 | 0.030068 | -0.042366 |
| | | 7 | 0.033841 | -0.040281 |
| | | 8 | 0.0377[illegible]2 | -0.037423 |
| | | 9 | 0.041903 | -0.033716 |
| | | 10 | 0.046286 | -0.029044 |
| | | 11 | 0.050984 | -0.023232 |
| | | 12 | 0.056083 | -0.016030 |
| | | 13 | 0.061695 | -0.007067 |
| | | 14 | 0.067989 | 0.004227 |
| | | 15 | 0.075216 | 0.018774 |
| | | 16 | 0.083802 | 0.039205 |
| | | 17 | 0.094526 | 0.065788 |
| | | 18 | 0.223648 | 0.317554 |
| E(LU) | | | 0.05998848 | |
| E(CP) | | | -0.00988868 | |
| E(LB) | | | | 0.03791809 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 19 | 19 | 1 | 0.014282 | -0.035995 |
| | | 2 | 0.018115 | -0.038600 |
| | | 3 | 0.021661 | -0.039833 |
| | | 4 | 0.025107 | -0.040204 |
| | | 5 | 0.028531 | -0.039873 |
| | | 6 | 0.031980 | -0.038897 |
| | | 7 | 0.035494 | -0.037282 |
| | | 8 | 0.039109 | -0.034997 |
| | | 9 | 0.042861 | -0.031977 |
| | | 10 | 0.046794 | -0.028121 |
| | | 11 | 0.050958 | -0.023280 |
| | | 12 | 0.055419 | -0.017234 |
| | | 13 | 0.060267 | -0.009660 |
| | | 14 | 0.065629 | -0.000055 |
| | | 15 | 0.071704 | 0.012400 |
| | | 16 | 0.078826 | 0.029177 |
| | | 17 | 0.087648 | 0.053305 |
| | | 18 | 0.099799 | 0.092832 |
| | | 19 | 0.125817 | 0.228292 |
| E(LU) | | | 0.05874886 | |
| E(CP) | | | -0.01213794 | |
| E(LB) | | | | 0.03383684 |
| 20 | 2 | 1 | -1.251068 | -0.493634 |
| | | 2 | 2.251068 | 0.493634 |
| E(LU) | | | 3.88005370 | |
| E(CP) | | | 1.26365389 | |
| E(LB) | | | | 0.49359782 |
| 20 | 3 | 1 | -0.662014 | -0.327555 |
| | | 2 | -0.621129 | -0.316157 |
| | | 3 | 2.283164 | 0.643713 |
| E(LU) | | | 1.75538518 | |
| E(CP) | | | 0.66462201 | |
| E(LB) | | | | 0.32470597 |
| 20 | 4 | 1 | -0.408252 | -0.243885 |
| | | 2 | -0.386289 | -0.238726 |
| | | 3 | -0.345972 | -0.223154 |
| | | 4 | 2.140513 | 0.705766 |
| E(LU) | | | 0.97791855 | |
| E(CP) | | | 0.40827717 | |
| E(LB) | | | | 0.24018443 |
| 20 | 5 | 1 | -0.272551 | -0.193344 |
| | | 2 | -0.259179 | -0.191385 |
| | | 3 | -0.233536 | -0.181278 |
| | | 4 | -0.199675 | -0.165821 |
| | | 5 | 1.964941 | 0.731828 |
| E(LU) | | | 0.61184794 | |
| E(CP) | | | 0.27193675 | |
| E(LB) | | | | 0.18940540 |
| 20 | 6 | 1 | -0.190502 | -0.159435 |
| | | 2 | -0.181539 | -0.159298 |
| | | 3 | -0.163969 | -0.152528 |
| | | 4 | -0.140605 | -0.141408 |
| | | 5 | -0.112303 | -0.126651 |
| | | 6 | 1.788917 | 0.739321 |
| E(LU) | | | 0.41329354 | |
| E(CP) | | | 0.18987852 | |
| E(LB) | | | | 0.15549251 |
| 20 | 7 | 1 | -0.136790 | -0.135060 |
| | | 2 | -0.130264 | -0.136029 |
| | | 3 | -0.117518 | -0.131448 |
| | | 4 | -0.100561 | -0.123236 |
| | | 5 | -0.079995 | -0.111990 |
| | | 6 | -0.056007 | -0.097918 |
| | | 7 | 1.621135 | 0.735681 |
| E(LU) | | | 0.29539488 | |
| E(CP) | | | 0.13637533 | |
| E(LB) | | | | 0.13121241 |
| 20 | 8 | 1 | -0.099621 | -0.116659 |
| | | 2 | -0.094504 | -0.118326 |
| | | 3 | -0.084808 | -0.115255 |
| | | 4 | -0.071993 | -0.109093 |
| | | 5 | -0.056488 | -0.100352 |
| | | 6 | -0.038416 | -0.089210 |
| | | 7 | -0.017755 | -0.075681 |
| | | 8 | 1.463595 | 0.724575 |
| E(LU) | | | 0.22087332 | |
| E(CP) | | | 0.09948206 | |
| E(LB) | | | | 0.11294771 |
| 20 | 9 | 1 | -0.072826 | -0.102246 |
| | | 2 | -0.068544 | -0.104362 |
| | | 3 | -0.060858 | -0.102371 |
| | | 4 | -0.050834 | -0.097711 |
| | | 5 | -0.038781 | -0.090828 |
| | | 6 | -0.024779 | -0.081874 |
| | | 7 | -0.008798 | -0.070863 |
| | | 8 | 0.009270 | -0.057714 |
| | | 9 | 1.316151 | 0.707969 |
| E(LU) | | | 0.17159045 | |
| E(CP) | | | 0.07297238 | |
| E(LB) | | | | 0.09868793 |
| 20 | 10 | 1 | -0.052900 | -0.090626 |
| | | 2 | -0.049115 | -0.093031 |
| | | 3 | -0.042792 | -0.091837 |
| | | 4 | -0.034710 | -0.088309 |
| | | 5 | -0.025087 | -0.082842 |
| | | 6 | -0.013973 | -0.075573 |
| | | 7 | -0.001335 | -0.066511 |
| | | 8 | 0.012921 | -0.055584 |
| | | 9 | 0.028939 | -0.042651 |
| | | 10 | 1.178052 | 0.686964 |
| E(LU) | | | 0.13788300 | |
| E(CP) | | | 0.05331635 | |
| E(LB) | | | | 0.08722579 |
| 20 | 11 | 1 | -0.037716 | -0.081036 |
| | | 2 | -0.034222 | -0.083625 |
| | | 3 | -0.028845 | -0.083028 |
| | | 4 | -0.022146 | -0.080373 |
| | | 5 | -0.014276 | -0.076014 |
| | | 6 | -0.005262 | -0.070070 |
| | | 7 | 0.004930 | -0.062554 |
| | | 8 | 0.016382 | -0.053398 |
| | | 9 | 0.029216 | -0.042476 |
| | | 10 | 0.043593 | -0.029594 |
| | | 11 | 1.048347 | 0.662168 |
| E(LU) | | | 0.11423656 | |
| E(CP) | | | 0.03838053 | |
| E(LB) | | | | 0.07779186 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | | C(N,M,I) |
|---|---|---|---|---|---|
| 20 | 12 | 1 | -0.025922 | | -0.072964 |
| | | 2 | -0.022589 | | -0.075662 |
| | | 3 | -0.017879 | | -0.075522 |
| | | 4 | -0.012183 | | -0.073554 |
| | | 5 | -0.005600 | | -0.070076 |
| | | 6 | 0.001858 | | -0.065197 |
| | | 7 | 0.010227 | | -0.058928 |
| | | 8 | 0.019578 | | -0.051211 |
| | | 9 | 0.030012 | | -0.041931 |
| | | 10 | 0.041668 | | -0.030912 |
| | | 11 | 0.054724 | | -0.017911 |
| | | 12 | 0.926107 | | 0.633868 |
| E(LU) | | | 0.09732994 | | |
| E(CP) | | | | 0.02680890 | |
| E(LB) | | | | | 0.06987174 |
| 20 | 13 | 1 | -0.016619 | | -0.066052 |
| | | 2 | -0.013364 | | -0.068809 |
| | | 3 | -0.009129 | | -0.069021 |
| | | 4 | -0.004170 | | -0.067601 |
| | | 5 | 0.001453 | | -0.064835 |
| | | 6 | 0.007742 | | -0.060825 |
| | | 7 | 0.014732 | | -0.055581 |
| | | 8 | 0.022485 | | -0.049051 |
| | | 9 | 0.031087 | | -0.041132 |
| | | 10 | 0.040654 | | -0.031666 |
| | | 11 | 0.051333 | | -0.020429 |
| | | 12 | 0.063321 | | -0.007116 |
| | | 13 | 0.810474 | | 0.602120 |
| E(LU) | | | 0.08507430 | | |
| E(C. ) | | | | 0.01770390 | |
| E(LB) | | | | | 0.06310742 |
| 20 | 14 | 1 | -0.009191 | | -0.060043 |
| | | 2 | -0.005961 | | -0.062821 |
| | | 3 | -0.002065 | | -0.063307 |
| | | 4 | 0.002348 | | -0.062329 |
| | | 5 | 0.007248 | | -0.060148 |
| | | 6 | 0.012649 | | -0.056856 |
| | | 7 | 0.018585 | | -0.052465 |
| | | 8 | 0.025109 | | -0.046929 |
| | | 9 | 0.032297 | | -0.040154 |
| | | 10 | 0.040241 | | -0.031999 |
| | | 11 | 0.049069 | | -0.022261 |
| | | 12 | 0.058941 | | -0.010659 |
| | | 13 | 0.070069 | | 0.003196 |
| | | 14 | 0.700660 | | 0.566775 |
| E(LU) | | | 0.07610847 | | |
| E(CP) | | | | 0.01045130 | |
| E(LB) | | | | | 0.05724068 |
| 20 | 15 | 1 | -0.003203 | | -0.054744 |
| | | 2 | 0.000035 | | -0.057513 |
| | | 3 | 0.003690 | | -0.058213 |
| | | 4 | 0.007695 | | -0.057596 |
| | | 5 | 0.012048 | | -0.055899 |
| | | 6 | 0.016769 | | -0.053209 |
| | | 7 | 0.021892 | | -0.049537 |
| | | 8 | 0.027467 | | -0.044842 |
| | | 9 | 0.033552 | | -0.039043 |
| | | 10 | 0.040230 | | -0.032010 |
| | | 11 | 0.047602 | | -0.023560 |
| | | 12 | 0.055804 | | -0.013436 |
| | | 13 | 0.065012 | | -0.001280 |
| | | 14 | 0.075467 | | 0.013416 |
| | | 15 | 0.595940 | | 0.527466 |
| E(LU) | | | 0.06951916 | | |
| E(CP) | | | | 0.00461910 | |
| E(LB) | | | | | 0.05207861 |
| 20 | 16 | 1 | 0.001656 | | -0.050002 |
| | | 2 | 0.004926 | | -0.052742 |
| | | 3 | 0.008410 | | -0.053608 |
| | | 4 | 0.012111 | | -0.053287 |
| | | 5 | 0.016048 | | -0.051997 |
| | | 6 | 0.020247 | | -0.049816 |
| | | 7 | 0.024742 | | -0.046757 |
| | | 8 | 0.029575 | | -0.042785 |
| | | 9 | 0.034801 | | -0.037825 |
| | | 10 | 0.040482 | | -0.031763 |
| | | 11 | 0.046707 | | -0.024432 |
| | | 12 | 0.053585 | | -0.015601 |
| | | 13 | 0.061262 | | -0.004939 |
| | | 14 | 0.069938 | | 0.008023 |
| | | 15 | 0.079893 | | 0.023984 |
| | | 16 | 0.495626 | | 0.483548 |
| E(LU) | | | 0.06467887 | | |
| E(CP) | | | | -0.00010333 | |
| E(LB) | | | | | 0.04747116 |
| 20 | 17 | 1 | 0.005617 | | -0.045695 |
| | | 2 | 0.008931 | | -0.048385 |
| | | 3 | 0.012297 | | -0.049380 |
| | | 4 | 0.015773 | | -0.049304 |
| | | 5 | 0.019394 | | -0.048357 |
| | | 6 | 0.023192 | | -0.046613 |
| | | 7 | 0.027201 | | -0.044083 |
| | | 8 | 0.031459 | | -0.040736 |
| | | 9 | 0.036010 | | -0.036510 |
| | | 10 | 0.040910 | | -0.031298 |
| | | 11 | 0.046228 | | -0.024954 |
| | | 12 | 0.052055 | | -0.017265 |
| | | 13 | 0.058509 | | -0.007934 |
| | | 14 | 0.065756 | | 0.003474 |
| | | 15 | 0.074029 | | 0.017606 |
| | | 16 | 0.083671 | | 0.035466 |
| | | 17 | 0.390968 | | 0.433947 |
| E(LU) | | | 0.06114864 | | |
| E(CP) | | | | -0.00394307 | |
| E(LB) | | | | | 0.04329477 |

| N | M | I | A(N,M,I) | | C(N,M,I) |
|---|---|---|---|---|---|
| 20 | 18 | 1 | 0.008847 | | -0.041706 |
| | | 2 | 0.012215 | | -0.044331 |
| | | 3 | 0.015502 | | -0.045472 |
| | | 4 | 0.018813 | | -0.045550 |
| | | 5 | 0.022197 | | -0.044896 |
| | | 6 | 0.025690 | | -0.043529 |
| | | 7 | 0.029324 | | -0.041460 |
| | | 8 | 0.033135 | | -0.038646 |
| | | 9 | 0.037162 | | -0.035086 |
| | | 10 | 0.041450 | | -0.030632 |
| | | 11 | 0.046055 | | -0.025168 |
| | | 12 | 0.051050 | | -0.018506 |
| | | 13 | 0.056532 | | -0.010374 |
| | | 14 | 0.062634 | | -0.000381 |
| | | 15 | 0.069547 | | 0.012071 |
| | | 16 | 0.077558 | | 0.027938 |
| | | 17 | 0.087131 | | 0.048871 |
| | | 18 | 0.305157 | | 0.376826 |
| E(LU) | | | 0.0[illegible]1867 | | |
| E(CP) | | | | -0.00706723 | |
| E(LB) | | | | | 0.03943688 |
| 20 | 19 | 1 | 0.011469 | | -0.037905 |
| | | 2 | 0.014895 | | -0.040446 |
| | | 3 | 0.018135 | | -0.041607 |
| | | 4 | 0.021329 | | -0.041903 |
| | | 5 | 0.024538 | | -0.041503 |
| | | 6 | 0.027802 | | -0.040467 |
| | | 7 | 0.031153 | | -0.038810 |
| | | 8 | 0.034624 | | -0.036509 |
| | | 9 | 0.038247 | | -0.033514 |
| | | 10 | 0.042061 | | -0.029746 |
| | | 11 | 0.046112 | | -0.025085 |
| | | 12 | 0.050458 | | -0.019364 |
| | | 13 | 0.055176 | | -0.012340 |
| | | 14 | 0.060372 | | -0.00[illegible]40 |
| | | 15 | 0.066198 | | 0.007217 |
| | | 16 | 0.072887 | | 0.021167 |
| | | 17 | 0.080830 | | 0.039737 |
| | | 18 | 0.090746 | | 0.066024 |
| | | 19 | 0.212971 | | 0.308714 |
| E(LU) | | | 0.05687410 | | |
| E(CP) | | | | -0.00959610 | |
| E(LB) | | | | | 0.03577112 |

| N | M | I | A(N,M,I) | | C(N,M,I) |
|---|---|---|---|---|---|
| 20 | 20 | 1 | 0.013553 | | -0.034055 |
| | | 2 | 0.017039 | | -0.036484 |
| | | 3 | 0.020257 | | -0.037686 |
| | | 4 | 0.023376 | | -0.038123 |
| | | 5 | 0.026464 | | -0.037945 |
| | | 6 | 0.029565 | | -0.037211 |
| | | 7 | 0.032711 | | -0.035932 |
| | | 8 | 0.035932 | | -0.034091 |
| | | 9 | 0.039258 | | -0.031646 |
| | | 10 | 0.042720 | | -0.028527 |
| | | 11 | 0.046357 | | -0.024632 |
| | | 12 | 0.050215 | | -0.019814 |
| | | 13 | 0.054354 | | -0.013860 |
| | | 14 | 0.058856 | | -0.006460 |
| | | 15 | 0.063842 | | 0.002865 |
| | | 16 | 0.069496 | | 0.014902 |
| | | 17 | 0.076128 | | 0.031052 |
| | | 18 | 0.084346 | | 0.054203 |
| | | 19 | 0.095669 | | 0.092028 |
| | | 20 | 0.119862 | | 0.221415 |
| E(LU) | | | 0.05578958 | | |
| E(CP) | | | | -0.01159947 | |
| E(LB) | | | | | 0.03207039 |
| 21 | 2 | 1 | -1.276880 | | -0.493942 |
| | | 2 | 2.276880 | | 0.493942 |
| E(LU) | | | 4.00827646 | | |
| E(CP) | | | | 1.28886132 | |
| E(LB) | | | | | 0.49390979 |
| 21 | 3 | 1 | -0.679427 | | -0.327841 |
| | | 2 | -0.640016 | | -0.316999 |
| | | 3 | 2.319443 | | 0.644841 |
| E(LU) | | | 1.82463798 | | |
| E(CP) | | | | 0.68177635 | |
| E(LB) | | | | | 0.32513088 |
| 21 | 4 | 1 | -0.421487 | | -0.244196 |
| | | 2 | -0.400349 | | -0.239279 |
| | | 3 | -0.361352 | | -0.224498 |
| | | 4 | 2.183188 | | 0.707973 |
| E(LU) | | | 1.02150585 | | |
| E(CP) | | | | 0.42133361 | |
| E(LB) | | | | | 0.24067351 |
| 21 | 5 | 1 | -0.283275 | | -0.193691 |
| | | 2 | -0.270454 | | -0.191814 |
| | | 3 | -0.245624 | | -0.182209 |
| | | 4 | -0.212831 | | -0.167572 |
| | | 5 | 2.012185 | | 0.735285 |
| E(LU) | | | 0.64156176 | | |
| E(CP) | | | | 0.28249590 | |
| E(LB) | | | | | 0.18993997 |
| 21 | 6 | 1 | -0.199541 | | -0.159820 |
| | | 2 | -0.191002 | | -0.159675 |
| | | 3 | -0.173984 | | -0.153231 |
| | | 4 | -0.151319 | | -0.142690 |
| | | 5 | -0.123892 | | -0.128756 |
| | | 6 | 1.839738 | | 0.744172 |
| E(LU) | | | 0.43452074 | | |
| E(CP) | | | | 0.19874801 | |
| E(LB) | | | | | 0.15606403 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 21 | 7 | 1 | -0.144615 | -0.135485 |
| | | 2 | -0.138450 | -0.136392 |
| | | 3 | -0.126112 | -0.132021 |
| | | 4 | -0.109643 | -0.124225 |
| | | 5 | -0.089672 | -0.113595 |
| | | 6 | -0.066419 | -0.100351 |
| | | 7 | 1.674911 | 0.742070 |
| E(LU) | | | 0.31099565 | |
| E(CP) | | | 0.14402019 | |
| E(LB) | | | | 0.13181685 |
| 21 | 8 | 1 | -0.106524 | -0.117124 |
| | | 2 | -0.101739 | -0.118696 |
| | | 3 | -0.092368 | -0.115755 |
| | | 4 | -0.079914 | -0.109895 |
| | | 5 | -0.064832 | -0.101621 |
| | | 6 | -0.047274 | -0.091123 |
| | | 7 | -0.027245 | -0.078439 |
| | | 8 | 1.519896 | 0.73[illegible]652 |
| E(LU) | | | 0.23252621 | |
| E(CP) | | | 0.10619473 | |
| E(LB) | | | | 0.11358345 |
| 21 | 9 | 1 | -0.079003 | -0.102752 |
| | | 2 | -0.075039 | -0.104752 |
| | | 3 | -0.067630 | -0.102836 |
| | | 4 | -0.057890 | -0.098393 |
| | | 5 | -0.046151 | -0.091865 |
| | | 6 | -0.032519 | -0.083417 |
| | | 7 | -0.016987 | -0.073081 |
| | | 8 | 0.000523 | -0.060807 |
| | | 9 | 1.374696 | 0.717905 |
| E(LU) | | | 0.18035373 | |
| E(CP) | | | 0.07894880 | |
| E(LB) | | | | 0.09935486 |
| 21 | 10 | 1 | -0.058485 | -0.091174 |
| | | 2 | -0.055015 | 0.0[illegible]3453 |
| | | 3 | -0.048942 | -0.092291 |
| | | 4 | -0.041036 | -0.088916 |
| | | 5 | -0.031718 | -0.083721 |
| | | 6 | -0.020880 | -0.076849 |
| | | 7 | -0.006567 | -0.068330 |
| | | 8 | 0.005290 | -0.058118 |
| | | 9 | 0.020804 | -0.046107 |
| | | 10 | 1.238610 | 0.698958 |
| E(LU) | | | 0.14446082 | |
| E(CP) | | | 0.05869415 | |
| E(LB) | | | | 0.08792500 |
| 21 | 11 | 1 | -0.042812 | -0.081628 |
| | | 2 | -0.039631 | -0.084084 |
| | | 3 | -0.034490 | -0.0834[illegible]9 |
| | | 4 | -0.027998 | .080939 |
| | | 5 | -0.020328 | -0.076784 |
| | | 6 | -0.011527 | -0.071153 |
| | | 7 | -0.001577 | -0.064073 |
| | | 8 | 0.009585 | -0.055502 |
| | | 9 | 0.022055 | -0.045346 |
| | | 10 | 0.035968 | -0.033457 |
| | | 11 | 1.110755 | 0.676455 |
| E(LU) | | | 0.11911746 | |
| E(CP) | | | 0.04325992 | |
| E(LB) | | | | 0.07852549 |
| 21 | 12 | 1 | -0.030601 | -0.073604 |
| | | 2 | -0.027584 | -0.076167 |
| | | 3 | -0.023103 | -0.076006 |
| | | 4 | -0.017596 | -0.074104 |
| | | 5 | -0.011187 | -0.070777 |
| | | 6 | -0.003901 | -0.066142 |
| | | 7 | 0.004282 | -0.060223 |
| | | 8 | 0.013418 | -0.052983 |
| | | 9 | 0.023589 | -0.044337 |
| | | 10 | 0.034909 | -0.034153 |
| | | 11 | 0.047525 | -0.022243 |
| | | 12 | 0.990248 | 0.65073[illegible] |
| E(LU) | | | 0.10086432 | |
| E(CP) | | | 0.03126495 | |
| E(LB) | | | | 0.07064304 |
| 21 | 13 | 1 | -0.020936 | -0.066744 |
| | | 2 | -0.018002 | -0.069367 |
| | | 3 | -0.013994 | -0.069541 |
| | | 4 | -0.009216 | -0.068156 |
| | | 5 | -0.003750 | -0.065499 |
| | | 6 | 0.002392 | -0.061675 |
| | | 7 | 0.009234 | -0.056708 |
| | | 8 | 0.016823 | -0.050566 |
| | | 9 | 0.025231 | -0.043172 |
| | | 10 | 0.034555 | -0.034404 |
| | | 11 | 0.044917 | -0.024094 |
| | | 12 | 0.056476 | -0.012009 |
| | | 13 | 0.876270 | 0.621934 |
| E(LU) | | | 0.08752028 | |
| E(CP) | | | 0.02179399 | |
| E(LB) | | | | 0.06392100 |
| 21 | 14 | 1 | -0.013190 | -0.060794 |
| | | 2 | -0.010287 | -0.063440 |
| | | 3 | -0.006618 | -0.063875 |
| | | 4 | -0.002385 | -0.062909 |
| | | 5 | 0.002367 | -0.060800 |
| | | 6 | 0.007636 | -0.057647 |
| | | 7 | 0.013446 | -0.053473 |
| | | 8 | 0.019842 | -0.048247 |
| | | 9 | 0.026883 | -0.041903 |
| | | 10 | 0.034651 | -0.034330 |
| | | 11 | 0.043251 | -0.025373 |
| | | 12 | 0.052814 | -0.014822 |
| | | 13 | 0.063513 | -0.002388 |
| | | 14 | 0.768077 | 0.590001 |
| E(LU) | | | 0.07766021 | |
| E(CP) | | | 0.01421994 | |
| E(LB) | | | | 0.05810297 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 21 | 15 | 1 | -0.006919 | -0.055563 |
| | | 2 | -0.004012 | -0.058205 |
| | | 3 | -0.000589 | -0.058845 |
| | | 4 | 0.003237 | -0.058219 |
| | | 5 | 0.007443 | -0.056566 |
| | | 6 | 0.012039 | -0.053974 |
| | | 7 | 0.0170[illegible] | -0.050466 |
| | | 8 | 0.02[illegible]5[illegible] | -0.046017 |
| | | 9 | 0.02[illegible]37 | -0.040565 |
| | | 10 | 0.03[illegible]35 | -0.034010 |
| | | 11 | 0.04[illegible]242 | -0.026215 |
| | | 12 | 0.050225 | -0.016982 |
| | | 13 | 0.059124 | -0.006049 |
| | | 14 | 0.069130 | 0.006947 |
| | | 15 | 0.664991 | 0.554729 |
| E(LU) | | | 0.07032505 | |
| E(CP) | | | 0.00810102 | |
| E(LB) | | | | 0.05299863 |
| 21 | 16 | 1 | -0.001802 | -0.050903 |
| | | 2 | 0.001131 | -0.053522 |
| | | 3 | 0.004378 | -0.054323 |
| | | 4 | 0.007896 | -0.053977 |
| | | 5 | 0.011684 | -0.052704 |
| | | 6 | 0.015759 | -0.050587 |
| | | 7 | 0.020147 | -0.047646 |
| | | 8 | 0.024883 | -0.043862 |
| | | 9 | 0.030017 | -0.039172 |
| | | 10 | 0.035594 | -0.033501 |
| | | 11 | 0.041700 | -0.026708 |
| | | 12 | 0.048420 | -0.018625 |
| | | 13 | 0.055879 | -0.009004 |
| | | 14 | 0.064233 | 0.002489 |
| | | 15 | 0.073697 | 0.016331 |
| | | 16 | 0.566382 | 0.515715 |
| E(LU) | | | 0.06485329 | |
| E(CP) | | | 0.00311876 | |
| E(LB) | | | | 0.04846207 |
| 21 | 17 | 1 | 0.002397 | -0.046698 |
| | | 2 | 0.005370 | -0.049278 |
| | | 3 | 0.008493 | -0.050203 |
| | | 4 | 0.011778 | -0.050089 |
| | | 5 | 0.015245 | -0.049137 |
| | | 6 | 0.018916 | -0.047426 |
| | | 7 | 0.022818 | -0.044972 |
| | | 8 | 0.026981 | -0.041760 |
| | | 9 | 0.031445 | -0.037741 |
| | | 10 | 0.036263 | -0.032831 |
| | | 11 | 0.041487 | -0.026922 |
| | | 12 | 0.047201 | -0.019846 |
| | | 13 | 0.053501 | -0.011386 |
| | | 14 | 0.060520 | -0.001230 |
| | | 15 | 0.068439 | 0.011065 |
| | | 16 | 0.077509 | 0.026154 |
| | | 17 | 0.471636 | 0.472299 |
| E(LU) | | | 0.06077952 | |
| E(CP) | | | -0.00096075 | |
| E(LB) | | | | 0.04437683 |
| 21 | 18 | 1 | 0.005853 | -0.042848 |
| | | 2 | 0.008874 | -0.045374 |
| | | 3 | 0.011911 | -0.046394 |
| | | 4 | 0.015024 | -0.046474 |
| | | 5 | 0.0182[illegible]5 | -0.045796 |
| | | 6 | 0.021603 | -0.044433 |
| | | 7 | 0.0251[illegible] | -0.042403 |
| | | 8 | 0.028836 | -0.0[illegible]9694 |
| | | 9 | 0.032780 | -0.036255 |
| | | 10 | 0.036985 | -0.032027 |
| | | 11 | 0.041512 | -0.026894 |
| | | 12 | 0.046418 | -0.020718 |
| | | 13 | 0.051789 | -0.013292 |
| | | 14 | 0.057729 | -0.004339 |
| | | 15 | 0.064392 | 0.006557 |
| | | 16 | 0.071987 | 0.020003 |
| | | 17 | 0.080826 | 0.036967 |
| | | 18 | 0.380111 | 0.423414 |
| E(LU) | | | 0.05777143 | |
| E(CP) | | | -0.00431152 | |
| E(LB) | | | | 0.04064434 |
| 21 | 19 | 1 | 0.008698 | -0.039258 |
| | | 2 | 0.011772 | -0.041717 |
| | | 3 | 0.014753 | -0.042808 |
| | | 4 | 0.017739 | -0.043048 |
| | | 5 | 0.020774 | -0.042605 |
| | | 6 | 0.023891 | -0.041546 |
| | | 7 | 0.027116 | -0.039889 |
| | | 8 | 0.030478 | -0.037622 |
| | | 9 | 0.034006 | -0.034708 |
| | | 10 | 0.037735 | -0.031080 |
| | | 11 | 0.041707 | -0.026648 |
| | | 12 | 0.045971 | -0.021282 |
| | | 13 | 0.050597 | -0.014797 |
| | | 14 | 0.055671 | -0.006936 |
| | | 15 | 0.061314 | 0.002673 |
| | | 16 | 0.067704 | 0.014599 |
| | | 17 | 0.075101 | 0.029744 |
| | | 18 | 0.083931 | 0.049671 |
| | | 19 | 0.291040 | 0.367257 |
| E(LU) | | | 0.05559063 | |
| E(CP) | | | -0.00706342 | |
| E(LB) | | | | 0.03717178 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | | C(N,M,I) |
|---|---|---|---|---|---|
| 21 | 20 | 1 | 0.011028 | | -0.035816 |
| | | 2 | 0.014157 | | -0.038193 |
| | | 3 | 0.017105 | | -0.039333 |
| | | 4 | 0.020000 | | -0.039706 |
| | | 5 | 0.022898 | | -0.039466 |
| | | 6 | 0.025833 | | -0.038676 |
| | | 7 | 0.028833 | | -0.037352 |
| | | 8 | 0.031923 | | -0.035486 |
| | | 9 | 0.035133 | | -0.033042 |
| | | 10 | 0.038489 | | -0.029966 |
| | | 11 | 0.042026 | | -0.026176 |
| | | 12 | 0.045787 | | -0.021555 |
| | | 13 | 0.049824 | | -0.015938 |
| | | 14 | 0.054209 | | -0.009097 |
| | | 15 | 0.059038 | | -0.000691 |
| | | 16 | 0.064453 | | 0.009794 |
| | | 17 | 0.070670 | | 0.023194 |
| | | 18 | 0.078048 | | 0.040976 |
| | | 19 | 0.087254 | | 0.066087 |
| | | 20 | 0.203293 | | 0.300441 |
| E(LU) | | | 0.05407008 | | |
| E(CP) | | | | -0.00931060 | |
| E(LB) | | | | | 0.03385072 |
| 21 | 21 | 1 | 0.012894 | | -0.032310 |
| | | 2 | 0.016080 | | -0.034581 |
| | | 3 | 0.019015 | | -0.035746 |
| | | 4 | 0.021851 | | -0.036228 |
| | | 5 | 0.024653 | | -0.036168 |
| | | 6 | 0.027458 | | -0.035622 |
| | | 7 | 0.030295 | | -0.034606 |
| | | 8 | 0.033186 | | -0.033114 |
| | | 9 | 0.036159 | | -0.031116 |
| | | 10 | 0.039235 | | -0.028564 |
| | | 11 | 0.042446 | | -0.025388 |
| | | 12 | 0.045824 | | -0.021484 |
| | | 13 | 0.049414 | | -0.016709 |
| | | 14 | 0.053270 | | -0.010860 |
| | | 15 | 0.057470 | | -0.003637 |
| | | 16 | 0.062124 | | 0.005419 |
| | | 17 | 0.067404 | | 0.017060 |
| | | 18 | 0.073603 | | 0.032626 |
| | | 19 | 0.081287 | | 0.054878 |
| | | 20 | 0.091876 | | 0.091146 |
| | | 21 | 0.114456 | | 0.215004 |
| E(LU) | | | 0.05311439 | | |
| E(CP) | | | | -0.01110585 | |
| E(LB) | | | | | 0.03047836 |
| 22 | 2 | 1 | -1.301440 | | -0.494222 |
| | | 2 | 2.301440 | | 0.494222 |
| E(LU) | | | 4.13275081 | | |
| E(CP) | | | | 1.31287156 | |
| E(LB) | | | | | 0.49419277 |
| 22 | 3 | 1 | -0.695987 | | -0.328100 |
| | | 2 | -0.657036 | | -0.317763 |
| | | 3 | 2.353023 | | 0.645863 |
| E(LU) | | | 1.89217469 | | |
| E(CP) | | | | 0.69810867 | |
| E(LB) | | | | | 0.32551588 |
| 22 | 4 | 1 | -0.434070 | | -0.244476 |
| | | 2 | -0.413690 | | -0.239780 |
| | | 3 | -0.375918 | | -0.225713 |
| | | 4 | 2.223678 | | 0.709969 |
| E(LU) | | | 1.06422125 | | |
| E(CP) | | | | 0.43376212 | |
| E(LB) | | | | | 0.24111609 |
| 22 | 5 | 1 | -0.293467 | | -0.194003 |
| | | 2 | -0.281150 | | -0.192202 |
| | | 3 | -0.257076 | | -0.183051 |
| | | 4 | -0.225276 | | -0.169152 |
| | | 5 | 2.056969 | | 0.738407 |
| E(LU) | | | 0.67084108 | | |
| E(CF) | | | | 0.29254724 | |
| E(LB) | | | | | 0.19042304 |
| 22 | 6 | 1 | -0.208130 | | -0.160166 |
| | | 2 | -0.199975 | | -0.160015 |
| | | 3 | -0.183470 | | -0.153867 |
| | | 4 | -0.161458 | | -0.143848 |
| | | 5 | -0.134845 | | -0.130649 |
| | | 6 | 1.887878 | | 0.748545 |
| E(LU) | | | 0.45557038 | | |
| E(CP) | | | | 0.20719220 | |
| E(LB) | | | | | 0.15657969 |
| 22 | 7 | 1 | -0.152049 | | -0.135866 |
| | | 2 | -0.146209 | | -0.136718 |
| | | 3 | -0.134250 | | -0.132539 |
| | | 4 | -0.118236 | | -0.125119 |
| | | 5 | -0.098823 | | -0.115041 |
| | | 6 | -0.076253 | | -0.102533 |
| | | 7 | 1.725819 | | 0.747815 |
| E(LU) | | | 0.32658258 | | |
| E(CP) | | | | 0.15130047 | |
| E(LB) | | | | | 0.13236123 |
| 22 | 8 | 1 | -0.113082 | | -0.117539 |
| | | 2 | -0.108593 | | -0.119026 |
| | | 3 | -0.099523 | | -0.116206 |
| | | 4 | -0.087407 | | -0.110619 |
| | | 5 | -0.072723 | | -0.102765 |
| | | 6 | -0.055645 | | -0.092840 |
| | | 7 | -0.036204 | | -0.080900 |
| | | 8 | 1.513176 | | 0.739897 |
| E(LU) | | | 0.24427569 | | |
| E(CP) | | | | 0.11258985 | |
| E(LB) | | | | | 0.11415483 |
| 22 | 9 | 1 | -0.084872 | | -0.103202 |
| | | 2 | -0.081190 | | -0.105100 |
| | | 3 | -0.074036 | | -0.103253 |
| | | 4 | -0.064560 | | -0.099008 |
| | | 5 | -0.053119 | | -0.092802 |
| | | 6 | -0.039835 | | -0.084805 |
| | | 7 | -0.024723 | | -0.075066 |
| | | 8 | -0.007729 | | -0.063556 |
| | | 9 | 1.430063 | | 0.726792 |
| E(LU) | | | 0.18929138 | | |
| E(CP) | | | | 0.08464553 | |
| E(LB) | | | | | 0.09995287 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | | C(N,M,I) |
|---|---|---|---|---|---|
| 22 | 10 | 1 | -0.063797 | | -0.091660 |
| | | 2 | -0.060601 | | -0.093825 |
| | | 3 | -0.054757 | | -0.092695 |
| | | 4 | -0.047129 | | -0.089463 |
| | | 5 | -0.037983 | | -0.084513 |
| | | 6 | -0.027405 | | -0.077998 |
| | | 7 | -0.015399 | | -0.069960 |
| | | 8 | -0.001917 | | -0.060373 |
| | | 9 | 0.013133 | | -0.049161 |
| | | 10 | 1.295854 | | 0.709648 |
| E(LU) | | | 0.15126968 | | |
| E(CP) | | | | 0.06382371 | |
| E(LB) | | | | | 0.08855022 |
| 22 | 11 | 1 | -0.047658 | | -0.082152 |
| | | 2 | -0.044752 | | -0.084487 |
| | | 3 | -0.039824 | | -0.083898 |
| | | 4 | -0.033522 | | -0.081446 |
| | | 5 | -0.026041 | | -0.077478 |
| | | 6 | -0.017442 | | -0.072129 |
| | | 7 | -0.007722 | | -0.065437 |
| | | 8 | 0.003165 | | -0.057379 |
| | | 9 | 0.015297 | | -0.047887 |
| | | 10 | 0.028784 | | -0.036846 |
| | | 11 | 1.169717 | | 0.689139 |
| E(LU) | | | 0.12427189 | | |
| E(CP) | | | | 0.04791797 | |
| E(LB) | | | | | 0.07917935 |
| 22 | 12 | 1 | -0.035052 | | -0.074167 |
| | | 2 | -0.032313 | | -0.076608 |
| | | 3 | -0.028036 | | -0.076431 |
| | | 4 | -0.022704 | | -0.074593 |
| | | 5 | -0.016456 | | -0.071406 |
| | | 6 | -0.009333 | | -0.066992 |
| | | 7 | -0.001327 | | -0.061386 |
| | | 8 | 0.007603 | | -0.054568 |
| | | 9 | 0.017525 | | -0.046475 |
| | | 10 | 0.028533 | | -0.037005 |
| | | 11 | 0.040749 | | -0.026013 |
| | | 12 | 1.050810 | | 0.665643 |
| E(LU) | | | 0.10470500 | | |
| E(CP) | | | | 0.03552320 | |
| E(LB) | | | | | 0.07132781 |
| 22 | 13 | 1 | -0.025047 | | -0.067350 |
| | | 2 | -0.022395 | | -0.069850 |
| | | 3 | -0.018588 | | -0.069993 |
| | | 4 | -0.013974 | | -0.068645 |
| | | 5 | -0.008653 | | -0.066089 |
| | | 6 | -0.002649 | | -0.062438 |
| | | 7 | 0.004050 | | [illegible] |
| | | 8 | 0.011483 | | -0.051924 |
| | | 9 | 0.019706 | | [illegible] |
| | | 10 | 0.028801 | | -0.036822 |
| | | 11 | 0.038870 | | -0.027293 |
| | | 12 | 0.050045 | | -0.016224 |
| | | 13 | 0.938350 | | 0.639340 |
| E(LU) | | | 0.09029856 | | |
| E(CP) | | | | 0.02570744 | |
| E(LB) | | | | | 0.06463988 |
| 22 | 14 | 1 | -0.017003 | | -0.061447 |
| | | 2 | -0.014385 | | -0.063972 |
| | | 3 | -0.010918 | | -0.064365 |
| | | 4 | -0.006845 | | -0.063413 |
| | | 5 | -0.002229 | | -0.061375 |
| | | 6 | 0.002917 | | -0.058353 |
| | | 7 | 0.008610 | | -0.054376 |
| | | 8 | 0.014879 | | -0.049432 |
| | | 9 | 0.021782 | | -0.043466 |
| | | 10 | 0.029381 | | -0.036397 |
| | | 11 | 0.037767 | | -0.028103 |
| | | 12 | 0.047049 | | -0.018422 |
| | | 13 | 0.057369 | | -0.007137 |
| | | 14 | 0.831624 | | 0.610259 |
| E(LU) | | | 0.07956563 | | |
| E(CP) | | | | 0.01783144 | |
| E(LB) | | | | | 0.05886035 |
| 22 | 15 | 1 | -0.010467 | | -0.056269 |
| | | 2 | -0.007849 | | -0.058795 |
| | | 3 | -0.004630 | | -0.059383 |
| | | 4 | -0.000964 | | -0.058755 |
| | | 5 | 0.003111 | | -0.057145 |
| | | 6 | 0.007593 | | -0.054649 |
| | | 7 | 0.012499 | | -0.051296 |
| | | 8 | 0.017862 | | -0.047069 |
| | | 9 | 0.023724 | | -0.041928 |
| | | 10 | 0.030145 | | -0.035792 |
| | | 11 | 0.037196 | | -0.028555 |
| | | 12 | 0.044975 | | -0.020066 |
| | | 13 | 0.053597 | | -0.010126 |
| | | 14 | 0.063218 | | 0.001536 |
| | | 15 | 0.729991 | | 0.578292 |
| E(LU) | | | 0.07150301 | | |
| E(CP) | | | | 0.01144432 | |
| E(LB) | | | | | 0.05380054 |
| 22 | 16 | 1 | -0.005110 | | -0.051672 |
| | | 2 | -0.002471 | | -0.054179 |
| | | 3 | 0.000569 | | -0.054921 |
| | | 4 | 0.003924 | | -0.054559 |
| | | 5 | 0.007582 | | -0.053308 |
| | | 6 | 0.011543 | | -0.051259 |
| | | 7 | 0.015838 | | -0.048429 |
| | | 8 | 0.020484 | | -0.044819 |
| | | 9 | 0.025520 | | -0.040385 |
| | | 10 | 0.031004 | | -0.035054 |
| | | 11 | 0.036998 | | -0.028725 |
| | | 12 | 0.043567 | | -0.021274 |
| | | 13 | 0.050828 | | -0.012503 |
| | | 14 | 0.058901 | | -0.002170 |
| | | 15 | 0.067958 | | 0.010067 |
| | | 16 | 0.632864 | | [illegible] |
| E(LU) | | | 0.06541674 | | |
| E(CP) | | | | 0.00622046 | |
| E(LB) | | | | | 0.04931689 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 22 | 17 | 1 | -0.000692 | -0.047541 |
| | | 2 | 0.001982 | -0.050015 |
| | | 3 | 0.004893 | -0.050879 |
| | | 4 | 0.008014 | -0.050736 |
| | | 5 | 0.011345 | -0.049789 |
| | | 6 | 0.014906 | -0.048115 |
| | | 7 | 0.018711 | -0.045743 |
| | | 8 | 0.022792 | -0.042661 |
| | | 9 | 0.027167 | -0.038846 |
| | | 10 | 0.031923 | -0.034195 |
| | | 11 | 0.037042 | -0.028684 |
| | | 12 | 0.042658 | -0.022124 |
| | | 13 | 0.048810 | -0.014389 |
| | | 14 | 0[illegible]55631 | -0.005227 |
| | | 15 | 0.[illegible]3253 | 0.005668 |
| | | 16 | 0.071871 | 0.018749 |
| | | 17 | 0.539692 | 0.504527 |
| E(LU) | | | 0.06081663 | |
| E(CP) | | | 0.00192009 | |
| E(LB) | | | | 0.04529672 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 22 | 18 | 1 | 0.002969 | -0.043785 |
| | | 2 | 0.005687 | -0.046214 |
| | | 3 | 0.008507 | -0.047171 |
| | | 4 | 0.011449 | -0.047212 |
| | | 5 | 0.014531 | -0.046521 |
| | | 6 | 0.017772 | -0.045174 |
| | | 7 | 0.021193 | -0.043196 |
| | | 8 | 0.024827 | -0.040573 |
| | | 9 | [illegible]28687 | -0.037286 |
| | | 10 | [illegible]32827 | -0.033268 |
| | | 11 | 0.037279 | -0.028441 |
| | | 12 | 0.042106 | -0.022691 |
| | | 13 | 0.047371 | -0.015865 |
| | | 14 | 0.053169 | -0.007753 |
| | | 15 | 0.059619 | 0.001940 |
| | | 16 | 0.066884 | 0.013632 |
| | | 17 | 0.075192 | 0.027940 |
| | | 18 | 0.449933 | 0.461640 |
| E(LU) | | | 0.05735061 | |
| E(CP) | | | -0.00163612 | |
| E(LB) | | | | 0.04164798 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 2[illegible] | 19 | 1 | 0.006009 | -0.040322 |
| | | 2 | 0.008777 | -0.042695 |
| | | 3 | 0.011534 | -0.043723 |
| | | 4 | 0.014342 | -0.043917 |
| | | 5 | 0.017230 | -0.043447 |
| | | 6 | 0.020225 | -0.042380 |
| | | 7 | 0.023341 | -0.040750 |
| | | 8 | 0.026624 | -0.038526 |
| | | 9 | 0.030059 | -0.035724 |
| | | 10 | 0.033736 | -0.032232 |
| | | 11 | 0.037624 | -0.028048 |
| | | 12 | 0.041834 | -0.023001 |
| | | 13 | 0.046370 | -0.017006 |
| | | 14 | 0.051344 | -0.009832 |
| | | 15 | 0.056833 | -0.001233 |
| | | 16 | 0.062987 | 0.009193 |
| | | 17 | 0.069992 | 0.022017 |
| | | 18 | 0.078134 | 0.038155 |
| | | 19 | 0.363007 | 0.413471 |
| E(LU) | | | 0.05476281 | |
| E(CP) | | | -0.00458357 | |
| E(LB) | | | | 0.03829068 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 22 | 20 | 1 | 0.008533 | -0.037073 |
| | | 2 | 0.011352 | -0.039379 |
| | | 3 | 0.014069 | -0.040459 |
| | | 4 | 0.016778 | -0.040780 |
| | | 5 | 0.019519 | -0.040500 |
| | | 6 | 0.022321 | -0.039681 |
| | | 7 | 0.025204 | -0.038351 |
| | | 8 | 0.028200 | -0.036496 |
| | | 9 | 0.031317 | -0.034104 |
| | | 10 | 0.034600 | -0.031120 |
| | | 11 | 0.038064 | -0.027481 |
| | | 12 | 0.041759 | -0.023097 |
| | | 13 | 0.045722 | -0.017840 |
| | | 14 | 0.050022 | -0.011535 |
| | | 15 | 0.054736 | -0.003933 |
| | | 16 | 0.059977 | 0.005317 |
| | | 17 | 0.065907 | 0.016757 |
| | | 18 | 0.072766 | 0.031243 |
| | | 19 | 0.080947 | 0.050259 |
| | | 20 | 0.278209 | 0.358253 |
| E(LU) | | | 0.05286705 | |
| E(CP) | | | -0.00702486 | |
| E(LB) | | | | 0.03514712 |

TABLE OF HEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 22 | 21 | 1 | 0.010614 | -0.033940 |
| | | 2 | 0.013486 | -0.036167 |
| | | 3 | 0.016181 | -0.037280 |
| | | 4 | 0.018819 | -0.037707 |
| | | 5 | 0.021451 | -0.037592 |
| | | 6 | 0.024107 | -0.036993 |
| | | 7 | 0.026810 | -0.035935 |
| | | 8 | 0.029588 | -0.034409 |
| | | 9 | 0.032449 | -0.032401 |
| | | 10 | 0.035435 | -0.029863 |
| | | 11 | 0.038552 | -0.026747 |
| | | 12 | 0.041849 | -0.022960 |
| | | 13 | 0.045351 | -0.018398 |
| | | 14 | 0.049118 | -0.012895 |
| | | 15 | 0.053206 | -0.006235 |
| | | 16 | 0.057712 | 0.001909 |
| | | 17 | 0.062764 | 0.012027 |
| | | 18 | 0.068562 | 0.024916 |
| | | 19 | 0.075443 | 0.041976 |
| | | 20 | 0.084022 | 0.065013 |
| | | 21 | 0.194480 | 0.292681 |
| E(LU) | | | 0.05153187 | |
| E(CP) | | | -0.00903424 | |
| E(LB) | | | | 0.03212313 |
| 22 | 22 | 1 | 0.01[illegible] | -0.030733 |
| | | 2 | 0.01[illegible] | -0.032859 |
| | | 3 | 0.01[illegible] | -0.033986 |
| | | 4 | 0.0205[illegible] | -0.034499 |
| | | 5 | 0.023056 | -0.034529 |
| | | 6 | 0.025607 | -0.034130 |
| | | 7 | 0.028178 | -0.033325 |
| | | 8 | 0.030794 | -0.032106 |
| | | 9 | 0.033464 | -0.030463 |
| | | 10 | 0.036226 | -0.028354 |
| | | 11 | 0.039082 | -0.025735 |
| | | 12 | 0.042076 | -0.022528 |
| | | 13 | 0.045226 | -0.018638 |
| | | 14 | 0.048580 | -0.013922 |
| | | 15 | 0.052185 | -0.008184 |
| | | 16 | 0.056116 | -0.001137 |
| | | 17 | 0.060475 | 0.007660 |
| | | 18 | 0.065424 | 0.018928 |
| | | 19 | 0.071237 | 0.033950 |
| | | 20 | 0.078445 | 0.055370 |
| | | 21 | 0.088381 | 0.090209 |
| | | 22 | 0.109526 | 0.209011 |
| E(LU) | | | 0.05068421 | |
| E(CP) | | | -0.01065184 | |
| E(LB) | | | | 0.02903621 |
| 23 | 2 | 1 | -1.324862 | -0.494478 |
| | | 2 | 2.324862 | 0.494478 |
| E(LU) | | | 4.25371176 | |
| E(CP) | | | 1.33579271 | |
| E(LB) | | | | 0.49445062 |
| 23 | 3 | 1 | -0.711775 | -0.328336 |
| | | 2 | -0.674982 | -0.318458 |
| | | 3 | 2.386757 | 0.646794 |
| E(LU) | | | 1.95807953 | |
| E(CP) | | | 0.71369343 | |
| E(LB) | | | | 0.32586633 |
| 23 | 4 | 1 | -0.446061 | -0.244730 |
| | | 2 | -0.426381 | -0.240237 |
| | | 3 | -0.389748 | -0.226817 |
| | | 4 | 2.262191 | 0.711785 |
| E(LU) | | | 1.10608838 | |
| E(CP) | | | 0.44561957 | |
| E(LB) | | | | 0.24151852 |
| 23 | 5 | 1 | -0.303177 | -0.194285 |
| | | 2 | -0.291322 | -0.192554 |
| | | 3 | -0.267953 | -0.183817 |
| | | 4 | -0.237079 | -0.170584 |
| | | 5 | 2.099531 | 0.741241 |
| E(LU) | | | 0.69967926 | |
| E(CP) | | | 0.30213653 | |
| E(LB) | | | | 0.19086173 |
| 23 | 6 | 1 | -0.216311 | -0.160479 |
| | | 2 | -0.208505 | -0.160324 |
| | | 3 | -0.192480 | -0.154445 |
| | | 4 | -0.171078 | -0.144898 |
| | | 5 | -0.145224 | -0.132362 |
| | | 6 | 1.933598 | 0.752500 |
| E(LU) | | | 0.47641869 | |
| E(CP) | | | 0.21524910 | |
| E(LB) | | | | 0.15704731 |
| 23 | 7 | 1 | -0.159128 | -0.136209 |
| | | 2 | -0.153582 | -0.137013 |
| | | 3 | -0.141975 | -0.133009 |
| | | 4 | -0.126391 | -0.125931 |
| | | 5 | -0.107499 | -0.116350 |
| | | 6 | -0.085566 | -0.104499 |
| | | 7 | 1.774142 | 0.753011 |
| E(LU) | | | 0.34212138 | |
| E(CP) | | | 0.15824839 | |
| E(LB) | | | | 0.13285412 |
| 23 | 8 | 1 | -0.119329 | -0.117912 |
| | | 2 | -0.115104 | -0.119324 |
| | | 3 | -0.106313 | -0.116615 |
| | | 4 | -0.094514 | -0.111278 |
| | | 5 | -0.080205 | -0.103803 |
| | | 6 | -0.063578 | -0.094391 |
| | | 7 | -0.044683 | -0.083112 |
| | | 8 | 1.623725 | 0.746435 |
| E(LU) | | | 0.25608069 | |
| E(CP) | | | 0.11869505 | |
| E(LB) | | | | 0.11467125 |
| 23 | 9 | 1 | -0.090463 | -0.103606 |
| | | 2 | -0.087031 | -0.105411 |
| | | 3 | -0.080[illegible] | -0.103629 |
| | | 4 | -0.070[illegible] | -0.099566 |
| | | 5 | -0.0597[illegible] | -0.093651 |
| | | 6 | -0.046768 | -0.086060 |
| | | 7 | -0.032050 | -0.076851 |
| | | 8 | -0.015537 | -0.066015 |
| | | 9 | 1.482569 | 0.734790 |
| E(LU) | | | 0.19835768 | |
| E(CP) | | | 0.09008641 | |
| E(LB) | | | | 0.10049225 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 23 | 10 | 1 | -0.068856 | -0.092095 |
| | | 2 | -0.065905 | -0.094156 |
| | | 3 | -0.060269 | -0.093058 |
| | | 4 | -0.052846 | -0.089957 |
| | | 5 | -0.043918 | -0.085232 |
| | | 6 | -0.033587 | -0.079038 |
| | | 7 | -0.021873 | -0.071429 |
| | | 8 | -0.008741 | -0.062395 |
| | | 9 | 0.005680 | -0.051880 |
| | | 10 | 1.350115 | 0.719241 |

E(LU) 0.15826056
E(CP) 0.06872565
E(LB) 0.08911283

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 23 | 11 | 1 | -0.052275 | -0.082619 |
| | | 2 | -0.049614 | -0.084845 |
| | | 3 | -0.044879 | -0.084262 |
| | | 4 | -0.038754 | -0.081902 |
| | | 5 | -0.031451 | -0.078106 |
| | | 6 | -0.023044 | -0.073012 |
| | | 7 | -0.013543 | -0.066668 |
| | | 8 | -0.002916 | -0.059066 |
| | | 9 | 0.008899 | -0.050154 |
| | | 10 | 0.021994 | -0.039847 |
| | | 11 | 1.225582 | 0.700481 |

E(LU) 0.12964842
E(CP) 0.05237239
E(LB) 0.07976613

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 23 | 12 | 1 | -0.039296 | -0.074667 |
| | | 2 | -0.036802 | -0.076996 |
| | | 3 | -0.032710 | -0.076807 |
| | | 4 | -0.027537 | -0.075031 |
| | | 5 | -0.021441 | -0.071973 |
| | | 6 | -0.014473 | -0.067761 |
| | | 7 | -0.006638 | -0.062438 |
| | | 8 | 0.002097 | -0.055995 |
| | | 9 | 0.011784 | -0.048387 |
| | | 10 | 0.022502 | -0.039535 |
| | | 11 | 0.034352 | -0.029328 |
| | | 12 | 1.108163 | 0.678918 |

E(LU) 0.10879863
E(CP) 0.03959874
E(LB) 0.07194033

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 23 | 13 | 1 | -0.028970 | -0.067886 |
| | | 2 | -0.026565 | -0.070274 |
| | | 3 | -0.022939 | -0.070390 |
| | | 4 | -0.018475 | -0.069079 |
| | | 5 | -0.013288 | -0.066620 |
| | | 6 | -0.007415 | -0.063126 |
| | | 7 | -0.000851 | -0.058638 |
| | | 8 | 0.006430 | -0.053149 |
| | | 9 | 0.014477 | -0.046619 |
| | | 10 | 0.023356 | -0.038975 |
| | | 11 | 0.033154 | -0.030114 |
| | | 12 | 0.043981 | -0.019898 |
| | | 13 | 0.997106 | 0.654769 |

E(LU) 0.09335412
E(CP) 0.02945680
E(LB) 0.06528043

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 23 | 14 | 1 | -0.020644 | -0.067022 |
| | | 2 | -0.013278 | -0.064436 |
| | | 3 | -0.014991 | -0.064792 |
| | | 4 | -0.011061 | -0.063857 |
| | | 5 | -0.006570 | -0.061888 |
| | | 6 | -0.001537 | -0.058966 |
| | | 7 | 0.004039 | -0.055193 |
| | | 8 | 0.010191 | -0.050500 |
| | | 9 | 0.016959 | -0.044870 |
| | | 10 | 0.024395 | -0.038243 |
| | | 11 | 0.032581 | -0.030518 |
| | | 12 | 0.041605 | -0.021572 |
| | | 13 | 0.051586 | -0.011236 |
| | | 14 | 0.891725 | 0.628112 |

E(LU) 0.08176810
E(CP) 0.02129585
E(LB) 0.05953202

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 23 | 15 | 1 | -0.013858 | -0.056886 |
| | | 2 | -0.011496 | -0.059304 |
| | | 3 | -0.008457 | -0.059848 |
| | | 4 | -0.004935 | -0.059221 |
| | | 5 | -0.000978 | -0.057655 |
| | | 6 | 0.003395 | -0.055253 |
| | | 7 | 0.008206 | -0.052040 |
| | | 8 | 0.013472 | -0.048017 |
| | | 9 | 0.019220 | -0.043159 |
| | | 10 | 0.025528 | -0.037385 |
| | | 11 | 0.032425 | -0.030636 |
| | | 12 | 0.040015 | -0.022775 |
| | | 13 | 0.048387 | -0.013657 |
| | | 14 | 0.057669 | -0.003074 |
| | | 15 | 0.791406 | 0.598911 |

E(LU) 0.07299467
E(CP) 0.01465639
E(LB) 0.05450749

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 23 | 16 | 1 | -0.008277 | -0.052337 |
| | | 2 | -0.005899 | -0.054740 |
| | | 3 | -0.003041 | -0.055433 |
| | | 4 | 0.000172 | -0.055058 |
| | | 5 | 0.003709 | -0.053835 |
| | | 6 | 0.007575 | -0.051846 |
| | | 7 | 0.011777 | -0.049129 |
| | | 8 | 0.016329 | -0.045688 |
| | | 9 | 0.021298 | -0.041465 |
| | | 10 | 0.026669 | -0.036455 |
| | | 11 | 0.032560 | -0.030526 |
| | | 12 | 0.038985 | -0.023615 |
| | | 13 | 0.046066 | -0.015549 |
| | | 14 | 0.053887 | -0.006157 |
| | | 15 | 0.062596 | 0.004820 |
| | | 16 | 0.695593 | 0.567014 |

E(LU) 0.06630856
E(CP) 0.00920619
E(LB) 0.05006475

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 23 | 17 | 1 | -0.003655 | -0.048263 |
| | | 2 | -0.001244 | -0.050639 |
| | | 3 | 0.001481 | -0.051447 |
| | | 4 | 0.004455 | -0.051283 |
| | | 5 | 0.007669 | -0.050345 |
| | | 6 | 0.011133 | -0.048710 |
| | | 7 | 0.014831 | -0.046438 |
| | | 8 | 0.018871 | -0.043448 |
| | | 9 | 0.023152 | -0.039831 |
| | | 10 | 0.027817 | -0.035444 |
| | | 11 | 0.032865 | -0.030257 |
| | | 12 | 0.038384 | -0.024144 |
| | | 13 | 0.044389 | -0.017027 |
| | | 14 | 0.051034 | -0.008672 |
| | | 15 | 0.058398 | 0.001120 |
| | | 16 | 0.066644 | 0.012681 |
| | | 17 | 0.603776 | 0.532147 |
| E(LU) | | | 0.06119601 | |
| E(CP) | | | | 0.00470017 |
| E(LB) | | | | 0.04609331 |
| 23 | 18 | 1 | 0.000194 | -0.044575 |
| | | 2 | 0.002646 | -0.046912 |
| | | 3 | 0.005275 | -0.047812 |
| | | 4 | 0.008068 | -0.047822 |
| | | 5 | 0.011028 | -0.047126 |
| | | 6 | 0.014162 | -0.045809 |
| | | 7 | 0.017515 | -0.043866 |
| | | 8 | 0.021048 | -0.041362 |
| | | 9 | 0.024872 | -0.038183 |
| | | 10 | 0.028875 | -0.034430 |
| | | 11 | 0.033363 | -0.029780 |
| | | 12 | 0.038034 | -0.024480 |
| | | 13 | 0.043231 | -0.018136 |
| | | 14 | 0.048884 | -0.010732 |
| | | 15 | 0.055157 | -0.001985 |
| | | 16 | 0.062143 | 0.008368 |
| | | 17 | 0.070035 | 0.020767 |
| | | 18 | 0.515472 | 0.493876 |
| E(LU) | | | 0.05728653 | |
| E(CP) | | | | 0.00095448 |
| E(LB) | | | | 0.04250454 |
| 23 | 19 | 1 | 0.003411 | -0.041199 |
| | | 2 | 0.005909 | -0.043487 |
| | | 3 | 0.008471 | -0.044457 |
| | | 4 | 0.011123 | -0.044615 |
| | | 5 | 0.013887 | -0.044125 |
| | | 6 | 0.016780 | -0.043060 |
| | | 7 | 0.019798 | -0.041470 |
| | | 8 | 0.022996 | -0.039317 |
| | | 9 | 0.026380 | -0.036600 |
| | | 10 | 0.030001 | -0.033248 |
| | | 11 | 0.033825 | -0.029295 |
| | | 12 | 0.037942 | -0.024577 |
| | | 13 | 0.042427 | -0.018980 |
| | | 14 | 0.047315 | -0.012379 |
| | | 15 | 0.052660 | -0.004606 |
| | | 16 | 0.058612 | 0.004662 |
| | | 17 | 0.065310 | 0.015808 |
| | | 18 | 0.072957 | 0.029414 |
| | | 19 | 0.430194 | 0.451531 |
| E(LU) | | | 0.05430916 | |
| E(CP) | | | | -0.00217056 |
| E(LB) | | | | 0.03922450 |
| 23 | 20 | 1 | 0.006104 | -0.038066 |
| | | 2 | 0.008651 | -0.040298 |
| | | 3 | 0.011168 | -0.041321 |
| | | 4 | 0.013716 | -0.041559 |
| | | 5 | 0.016325 | -0.041290 |
| | | 6 | 0.019011 | -0.040466 |
| | | 7 | 0.021809 | -0.039131 |
| | | 8 | 0.024700 | -0.037336 |
| | | 9 | 0.027785 | -0.034966 |
| | | 10 | 0.030958 | -0.032134 |
| | | 11 | 0.034420 | -0.028603 |
| | | 12 | 0.037984 | -0.024527 |
| | | 13 | 0.041940 | -0.019546 |
| | | 14 | 0.046153 | -0.013730 |
| | | 15 | 0.050775 | -0.006799 |
| | | 16 | 0.055861 | 0.001462 |
| | | 17 | 0.061571 | 0.011459 |
| | | 18 | 0.068058 | 0.023716 |
| | | 19 | 0.075590 | 0.039106 |
| | | 20 | 0.347421 | 0.404070 |
| E(LU) | | | 0.05206387 | |
| E(CP) | | | | -0.00478196 |
| E(LB) | | | | 0.03618730 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 23 | 21 | 1 | 0.008356 | 0.[illegible]35111 |
| | | 2 | 0.010952 | -0.037277 |
| | | 3 | 0.013441 | -0.038337 |
| | | 4 | 0.015912 | -0.038[illegible]18 |
| | | 5 | 0.018403 | -0.038563 |
| | | 6 | 0.020936 | -0.037940 |
| | | 7 | 0.023539 | -0.036861 |
| | | 8 | 0.026213 | -0.035349 |
| | | 9 | 0.029021 | -0.033345 |
| | | 10 | 0.031907 | -0.030890 |
| | | 11 | 0.034969 | -0.027882 |
| | | 12 | 0.038215 | -0.024224 |
| | | 13 | 0.041650 | -0.019928 |
| | | 14 | 0.045352 | -0.014781 |
| | | 15 | 0.049362 | -0.008853 |
| | | 16 | 0.053758 | -0.001297 |
| | | 17 | 0.058642 | 0.007615 |
| | | 18 | 0.064166 | 0.018608 |
| | | 19 | 0.070550 | 0.032491 |
| | | 20 | 0.078160 | 0.050676 |
| | | 21 | 0.266495 | 0.349764 |
| E(LU) | | | 0.05040338 | |
| E(CP) | | | -0.00696129 | |
| E(LB) | | | | 0.03332701 |
| 23 | 22 | 1 | 0.010228 | -0.032246 |
| | | 2 | 0.012873 | -0.034337 |
| | | 3 | 0.015348 | -0.035418 |
| | | 4 | 0.017763 | -0.035883 |
| | | 5 | 0.020166 | -0.035864 |
| | | 6 | 0.022582 | -0.035420 |
| | | 7 | 0.025037 | -0.034566 |
| | | 8 | 0.027534 | -0.033327 |
| | | 9 | 0.030133 | -0.031642 |
| | | 10 | 0.032777 | -0.029557 |
| | | 11 | 0.035580 | -0.026947 |
| | | 12 | 0.038475 | -0.023825 |
| | | 13 | 0.041565 | -0.020057 |
| | | 14 | 0.044837 | -0.015569 |
| | | 15 | 0.048365 | -0.010179 |
| | | 16 | 0.052188 | -0.003701 |
| | | 17 | 0.056406 | 0.004192 |
| | | 18 | 0.061135 | 0.013968 |
| | | 19 | 0.066561 | 0.026384 |
| | | 20 | 0.073001 | 0.042777 |
| | | 21 | 0.081024 | 0.065832 |
| | | 22 | 0.186420 | 0.285386 |
| E(LU) | | | 0.04922310 | |
| E(CP) | | | -0.00876815 | |
| E(LB) | | | | 0.03056093 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 23 | 23 | 1 | 0.011749 | -0.029300 |
| | | 2 | 0.014443 | -0.031296 |
| | | 3 | 0.016915 | -0.032383 |
| | | 4 | 0.019296 | -0.032915 |
| | | 5 | 0.021637 | -0.033014 |
| | | 6 | 0.023968 | -0.032735 |
| | | 7 | 0.026314 | -0.032093 |
| | | 8 | 0.028683 | -0.031103 |
| | | 9 | 0.031120 | -0.029730 |
| | | 10 | 0.033589 | -0.027985 |
| | | 11 | 0.036174 | -0.025797 |
| | | 12 | 0.038830 | -0.023139 |
| | | 13 | 0.041635 | -0.019922 |
| | | 14 | 0.044584 | -0.016060 |
| | | 15 | 0.047728 | -0.011412 |
| | | 16 | 0.051110 | -0.005790 |
| | | 17 | 0.054801 | 0.001083 |
| | | 18 | 0.058897 | 0.009633 |
| | | 19 | 0.063548 | 0.020550 |
| | | 20 | 0.069018 | 0.035063 |
| | | 21 | 0.075800 | 0.055713 |
| | | 22 | 0.085149 | 0.089235 |
| | | 23 | 0.105012 | 0.203397 |
| E(LU) | | | 0.04846684 | |
| E(CP) | | | -0.01023295 | |
| E(LB) | | | | 0.02772378 |
| 24 | 2 | 1 | -1.347247 | -0.494711 |
| | | 2 | 2.347247 | 0.494711 |
| E(LU) | | | 4.37137282 | |
| E(CP) | | | 1.35771886 | |
| E(LB) | | | | 0.49468654 |
| 24 | 3 | 1 | -0.726858 | -0.328551 |
| | | 2 | -0.691234 | -0.319094 |
| | | 3 | 2.418093 | 0.647646 |
| E(LU) | | | 2.02243176 | |
| E(CP) | | | 0.72859553 | |
| E(LB) | | | | 0.32618670 |
| 24 | 4 | 1 | -0.457513 | -0.244962 |
| | | 2 | -0.438482 | -0.240654 |
| | | 3 | -0.402913 | -0.227825 |
| | | 4 | 2.298908 | 0.713442 |
| E(LU) | | | 1.14713265 | |
| E(CP) | | | 0.45695549 | |
| E(LB) | | | | 0.24198602 |
| 24 | 5 | 1 | -0.312447 | -0.194542 |
| | | 2 | -0.301019 | -0.192877 |
| | | 3 | -0.278309 | -0.184517 |
| | | 4 | -0.248301 | -0.171890 |
| | | 5 | 2.140076 | 0.743825 |
| E(LU) | | | 0.72807462 | |
| E(CP) | | | 0.31130369 | |
| E(LB) | | | | 0.19126190 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 24 | 6 | 1 | -0.224120 | -0.160763 |
| | | 2 | -0.216634 | -0.160605 |
| | | 3 | -0.201057 | -0.154973 |
| | | 4 | -0.180228 | -0.145857 |
| | | 5 | -0.155085 | -0.133918 |
| | | 6 | 1.977123 | 0.756116 |
| E(LU) | | | 0.49704860 | |
| E(CP) | | | 0.22295192 | |
| E(LB) | | | | 0.15747334 |
| 24 | 7 | 1 | -0.165886 | -0.136519 |
| | | 2 | -0.160606 | -0.137280 |
| | | 3 | -0.149328 | -0.133438 |
| | | 4 | -0.134146 | -0.126672 |
| | | 5 | -0.115746 | -0.117541 |
| | | 6 | -0.094409 | -0.106282 |
| | | 7 | 1.820121 | 0.757733 |
| E(LU) | | | 0.35758552 | |
| E(CP) | | | 0.16489218 | |
| E(LB) | | | | 0.13330255 |
| 24 | 8 | 1 | -0.125290 | -0.118250 |
| | | 2 | -0.121304 | -0.119593 |
| | | 3 | -0.112772 | -0.116987 |
| | | 4 | -0.101272 | -0.111878 |
| | | 5 | -0.087317 | -0.104747 |
| | | 6 | -0.071114 | -0.095799 |
| | | 7 | -0.052731 | -0.085111 |
| | | 8 | 1.671800 | 0.752364 |
| E(LU) | | | 0.26790847 | |
| E(CP) | | | 0.12453462 | |
| E(LB) | | | | 0.11514035 |
| 24 | 9 | 1 | -0.095798 | -0.103970 |
| | | 2 | -0.092592 | -0.105691 |
| | | 3 | -0.085890 | -0.103970 |
| | | 4 | -0.076896 | -0.100075 |
| | | 5 | -0.066000 | -0.094425 |
| | | 6 | -0.053356 | -0.087200 |
| | | 7 | -0.039009 | -0.078467 |
| | | 8 | -0.022943 | -0.068229 |
| | | 9 | 1.532484 | 0.742027 |
| E(LU) | | | 0.20751570 | |
| E(CP) | | | 0.09529249 | |
| E(LB) | | | | 0.10098133 |
| 24 | 10 | 1 | -0.073685 | -0.092486 |
| | | 2 | -0.070954 | -0.094454 |
| | | 3 | -0.065509 | -0.093387 |
| | | 4 | -0.058278 | -0.090406 |
| | | 5 | -0.049557 | -0.085886 |
| | | 6 | -0.039461 | -0.079984 |
| | | 7 | -0.028021 | -0.072761 |
| | | 8 | -0.015218 | -0.064218 |
| | | 9 | -0.000998 | -0.054317 |
| | | 10 | 1.401679 | 0.727899 |
| E(LU) | | | 0.16539357 | |
| E(CP) | | | 0.07341826 | |
| E(LB) | | | | 0.08962194 |
| 24 | 11 | 1 | -0.056685 | -0.083038 |
| | | 2 | -0.054241 | -0.085164 |
| | | 3 | -0.049681 | -0.084589 |
| | | 4 | -0.043721 | -0.082316 |
| | | 5 | -0.036586 | -0.078677 |
| | | 6 | -0.028363 | -0.073816 |
| | | 7 | -0.019070 | -0.067786 |
| | | 8 | -0.008690 | -0.060589 |
| | | 9 | 0.002829 | -0.052190 |
| | | 10 | 0.015559 | -0.042523 |
| | | 11 | 1.278649 | 0.710688 |
| E(LU) | | | 0.13520496 | |
| E(CP) | | | 0.05663907 | |
| E(LB) | | | | 0.08029586 |
| 24 | 12 | 1 | -0.043350 | -0.075114 |
| | | 2 | -0.041075 | -0.077341 |
| | | 3 | -0.037150 | -0.077143 |
| | | 4 | -0.032125 | -0.075425 |
| | | 5 | -0.026171 | -0.072488 |
| | | 6 | -0.019351 | -0.068461 |
| | | 7 | -0.011678 | -0.063393 |
| | | 8 | -0.003130 | -0.057285 |
| | | 9 | 0.006334 | -0.050108 |
| | | 10 | 0.016780 | -0.041798 |
| | | 11 | 0.028293 | -0.032267 |
| | | 12 | 1.162621 | 0.690824 |
| E(LU) | | | 0.11310142 | |
| E(CP) | | | 0.04350525 | |
| E(LB) | | | | 0.07249181 |
| 24 | 13 | 1 | -0.032719 | -0.068363 |
| | | 2 | -0.030535 | -0.070648 |
| | | 3 | -0.027071 | -0.070743 |
| | | 4 | -0.022744 | -0.069468 |
| | | 5 | -0.017683 | -0.067099 |
| | | 6 | -0.011934 | -0.063752 |
| | | 7 | -0.005499 | -0.059470 |
| | | 8 | 0.001635 | -0.054260 |
| | | 9 | 0.009515 | -0.048088 |
| | | 10 | 0.018189 | -0.040903 |
| | | 11 | 0.027735 | -0.032622 |
| | | 12 | 0.038242 | -0.023135 |
| | | 13 | 1.052869 | 0.668551 |
| E(LU) | | | 0.09664182 | |
| E(CP) | | | 0.03305372 | |
| E(LB) | | | | 0.06585529 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 24 | 14 | 1 | -0.024127 | -0.062531 |
| | | 2 | -0.021963 | -0.064844 |
| | | 3 | -0.018857 | -0.065168 |
| | | 4 | -0.015060 | -0.064253 |
| | | 5 | -0.010682 | -0.062346 |
| | | 6 | -0.005761 | -0.059561 |
| | | 7 | -0.000290 | -0.055934 |
| | | 8 | 0.005750 | -0.051466 |
| | | 9 | 0.012378 | -0.046144 |
| | | 10 | 0.019668 | -0.039899 |
| | | 11 | 0.027663 | -0.032671 |
| | | 12 | 0.036447 | -0.024354 |
| | | 13 | 0.046122 | -0.014814 |
| | | 14 | 0.948731 | 0.643985 |
| E(LU) | | | 0.08422122 | |
| E(CP) | | | 0.02462280 | |
| E(LB) | | | | 0.06013250 |
| 24 | 15 | 1 | -0.017105 | -0.057431 |
| | | 2 | -0.014969 | -0.059749 |
| | | 3 | -0.012092 | -0.060254 |
| | | 4 | -0.008696 | -0.059630 |
| | | 5 | -0.004850 | -0.058110 |
| | | 6 | -0.000583 | -0.055800 |
| | | 7 | 0.004161 | -0.052701 |
| | | 8 | 0.009290 | -0.048895 |
| | | 9 | 0.014974 | -0.044259 |
| | | 10 | 0.021148 | -0.038824 |
| | | 11 | 0.027900 | -0.032498 |
| | | 12 | 0.035313 | -0.025177 |
| | | 13 | 0.043457 | -0.016750 |
| | | 14 | 0.052436 | -0.007059 |
| | | 15 | 0.849616 | 0.617138 |
| E(LU) | | | 0.07475231 | |
| E(CP) | | | 0.01774484 | |
| E(LB) | | | | 0.05513653 |
| 24 | 16 | 1 | -0.011313 | -0.052921 |
| | | 2 | -0.009163 | -0.055227 |
| | | 3 | -0.006469 | -0.055875 |
| | | 4 | -0.003385 | -0.055494 |
| | | 5 | 0.000046 | -0.054297 |
| | | 6 | 0.003828 | -0.052365 |
| | | 7 | 0.007921 | -0.04977[illegible] |
| | | 8 | 0.012411 | -0.046465 |
| | | 9 | 0.017303 | -0.042445 |
| | | 10 | 0.022558 | -0.0377[illegible]3 |
| | | 11 | 0.028368 | -0.032133 |
| | | 12 | 0.034638 | -0.025703 |
| | | 13 | 0.041559 | -0.018228 |
| | | 14 | 0.049152 | -0.009617 |
| | | 15 | 0.057555 | 0.000342 |
| | | 16 | 0.754990 | 0.587928 |
| E(LU) | | | 0.06747956 | |
| E(CP) | | | 0.01208139 | |
| E(LB) | | | | 0.05072627 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 24 | 17 | 1 | -0.006499 | -0.048891 |
| | | 2 | -0.004321 | -0.051174 |
| | | 3 | -0.001761 | -0.051934 |
| | | 4 | 0.001076 | -0.051760 |
| | | 5 | 0.004211 | -0.050811 |
| | | 6 | 0.007533 | -0.049264 |
| | | 7 | 0.011242 | -0.046993 |
| | | 8 | 0.015072 | -0.044237 |
| | | 9 | 0.019386 | -0.040701 |
| | | 10 | 0.023957 | -0.036555 |
| | | 11 | 0.028939 | -0.031656 |
| | | 12 | 0.034306 | -0.025980 |
| | | 13 | 0.04020[illegible] | -0.019361 |
| | | 14 | 0.0467[illegible] | -0.011667 |
| | | 15 | 0.053823 | -0.002782 |
| | | 16 | 0.061755 | 0.007593 |
| | | 17 | 0.664373 | 0.556173 |
| E(LU) | | | 0.06186649 | |
| E(CP) | | | 0.00738246 | |
| E(LB) | | | | 0.04679261 |
| 24 | 18 | 1 | -0.002475 | -0.045255 |
| | | 2 | -0.000259 | -0.047504 |
| | | 3 | 0.002201 | -0.048354 |
| | | 4 | 0.004666 | -0.048336 |
| | | 5 | 0.007700 | -0.047659 |
| | | 6 | 0.010794 | -0.046318 |
| | | 7 | 0.014016 | -0.044487 |
| | | 8 | 0.017417 | -0.042119 |
| | | 9 | 0.021386 | -0.038895 |
| | | 10 | 0.025182 | -0.035448 |
| | | 11 | 0.029642 | -0.031020 |
| | | 12 | 0.034054 | -0.026209 |
| | | 13 | 0.039415 | -0.020075 |
| | | 14 | 0.044838 | -0.013352 |
| | | 15 | 0.050958 | -0.005370 |
| | | 16 | 0.057699 | 0.003929 |
| | | 17 | 0.065252 | 0.014885 |
| | | 18 | 0.577315 | 0.521587 |
| E(LU) | | | 0.05752515 | |
| E(CP) | | | 0.00346018 | |
| E(LB) | | | | 0.04324895 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 24 | 19 | 1 | 0.000905 | -0.041941 |
| | | 2 | 0.003164 | -0.044148 |
| | | 3 | 0.005555 | -0.045066 |
| | | 4 | 0.008071 | -0.045193 |
| | | 5 | 0.010734 | -0.044684 |
| | | 6 | 0.013521 | -0.043644 |
| | | 7 | 0.016435 | -0.042115 |
| | | 8 | 0.019637 | -0.039942 |
| | | 9 | 0.022803 | -0.037505 |
| | | 10 | 0.026719 | -0.033941 |
| | | 11 | 0.029997 | -0.030672 |
| | | 12 | 0.034402 | -0.025868 |
| | | 13 | 0.038730 | -0.020747 |
| | | 14 | 0.043507 | -0.014657 |
| | | 15 | 0.048732 | -0.007553 |
| | | 16 | 0.054520 | 0.000812 |
| | | 17 | 0.060958 | 0.010675 |
| | | 18 | 0.068216 | 0.022447 |
| | | 19 | 0.493393 | 0.48374[illegible] |
| E(LU) | | | 0.05417052 | |
| E(CP) | | | 0.00017117 | |
| E(LB) | | | | 0.04002428 |
| 24 | 20 | 1 | 0.003753 | -0.038888 |
| | | 2 | 0.006057 | -0.041046 |
| | | 3 | 0.008399 | -0.042017 |
| | | 4 | 0.010808 | -0.042259 |
| | | 5 | 0.013294 | -0.041940 |
| | | 6 | 0.015900 | -0.041093 |
| | | 7 | 0.018611 | -0.039782 |
| | | 8 | 0.021436 | -0.038014 |
| | | 9 | 0.024426 | -0.035765 |
| | | 10 | 0.027521 | -0.033081 |
| | | 11 | 0.031146 | -0.029441 |
| | | 12 | 0.034308 | -0.025968 |
| | | 13 | 0.038424 | -0.021075 |
| | | 14 | 0.042551 | -0.015682 |
| | | 15 | 0.047082 | -0.009322 |
| | | 16 | 0.052019 | -0.001870 |
| | | 17 | 0.057545 | 0.007016 |
| | | 18 | 0.063743 | 0.017651 |
| | | 19 | 0.070809 | 0.030651 |
| | | 20 | 0.412170 | 0.441924 |
| E(LU) | | | 0.05159088 | |
| E(CP) | | | -0.00259469 | |
| E(LB) | | | | 0.03705876 |
| 24 | 21 | 1 | 0.006154 | -0.036040 |
| | | 2 | 0.008506 | -0.038142 |
| | | 3 | 0.010815 | -0.039151 |
| | | 4 | 0.013140 | -0.039493 |
| | | 5 | 0.015514 | -0.039306 |
| | | 6 | 0.017945 | -0.038667 |
| | | 7 | 0.020429 | -0.037625 |
| | | 8 | 0.023099 | -0.036040 |
| | | 9 | 0.025743 | -0.034203 |
| | | 10 | 0.028697 | -0.031687 |
| | | 11 | 0.031580 | -0.028925 |
| | | 12 | 0.034836 | -0.025342 |
| | | 13 | 0.038156 | -0.021392 |
| | | 14 | 0.041933 | -0.016415 |
| | | 15 | 0.045799 | -0.010844 |
| | | 16 | 0.050108 | -0.004137 |
| | | 17 | 0.054842 | 0.003809 |
| | | 18 | 0.060165 | 0.013408 |
| | | 19 | 0.066188 | 0.025170 |
| | | 20 | 0.073197 | 0.039844 |
| | | 21 | 0.333154 | 0.395179 |
| E(LU) | | | 0.04962788 | |
| E(CP) | | | -0.00492314 | |
| E(LB) | | | | 0.03429681 |
| 24 | 22 | 1 | 0.008175 | -0.033339 |
| | | 2 | 0.010574 | -0.035378 |
| | | 3 | 0.012864 | -0.036413 |
| | | 4 | 0.015129 | -0.036835 |
| | | 5 | 0.017403 | -0.036782 |
| | | 6 | 0.019717 | -0.036300 |
| | | 7 | 0.022047 | -0.035464 |
| | | 8 | 0.024510 | -0.034155 |
| | | 9 | 0.026977 | -0.032554 |
| | | 10 | 0.029650 | -0.030414 |
| | | 11 | 0.032257 | -0.028021 |
| | | 12 | 0.035139 | -0.024936 |
| | | 13 | 0.038322 | -0.021171 |
| | | 14 | 0.041338 | -0.017209 |
| | | 15 | 0.044935 | -0.011998 |
| | | 16 | 0.048628 | -0.006114 |
| | | 17 | 0.052763 | 0.001032 |
| | | 18 | 0.057327 | 0.009615 |
| | | 19 | 0.062480 | 0.020215 |
| | | 20 | 0.068458 | 0.033512 |
| | | 21 | 0.075549 | 0.050964 |
| | | 22 | 0.255757 | 0.341745 |
| E(LU) | | | 0.04816352 | |
| E(CP) | | | -0.00687983 | |
| E(LB) | | | | 0.03168[illegible] |

TABLE OF WEIGHTS (CONTINUED)

| | N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|---|
| | 24 | 23 | 1 | 0.009865 | -0.030709 |
| | | | 2 | 0.012311 | -0.032676 |
| | | | 3 | 0.014593 | -0.033724 |
| | | | 4 | 0.016814 | -0.034214 |
| | | | 5 | 0.019018 | -0.034270 |
| | | | 6 | 0.021232 | -0.033942 |
| | | | 7 | 0.023451 | -0.033279 |
| | | | 8 | 0.025757 | -0.032215 |
| | | | 9 | 0.028049 | -0.030887 |
| | | | 10 | 0.030536 | -0.029034 |
| | | | 11 | 0.032941 | -0.026957 |
| | | | 12 | 0.035565 | -0.024275 |
| | | | 13 | 0.038329 | -0.021160 |
| | | | 14 | 0.041191 | -0.017439 |
| | | | 15 | 0.044281 | -0.013616 |
| | | | 16 | 0.047572 | -0.007758 |
| | | | 17 | 0.051172 | -0.001443 |
| | | | 18 | 0.055132 | 0.006201 |
| | | | 19 | 0.059561 | 0.015675 |
| | | | 20 | 0.064672 | 0.027621 |
| | | | 21 | 0.070703 | 0.043424 |
| | | | 22 | 0.078235 | 0.065565 |
| | | | 23 | 0.179020 | 0.278512 |
| E(LU) | | | | 0.04721383 | |
| E(CP) | | | | -0.00851289 | |
| E(LB) | | | | | 0.02914162 |
| | 24 | 24 | 1 | 0.011248 | -0.027993 |
| | | | 2 | 0.013739 | -0.029871 |
| | | | 3 | 0.016021 | -0.030917 |
| | | | 4 | 0.018216 | -0.031460 |
| | | | 5 | 0.020370 | -0.031614 |
| | | | 6 | 0.022514 | -0.031426 |
| | | | 7 | 0.024647 | -0.030730 |
| | | | 8 | 0.026838 | -0.030091 |
| | | | 9 | 0.029015 | -0.028989 |
| | | | 10 | 0.031329 | -0.027477 |
| | | | 11 | 0.033589 | -0.025684 |
| | | | 12 | 0.035989 | -0.023440 |
| | | | 13 | 0.038543 | -0.020740 |
| | | | 14 | 0.041131 | -0.017557 |
| | | | 15 | 0.043928 | -0.013710 |
| | | | 16 | 0.046862 | -0.009153 |
| | | | 17 | 0.050034 | -0.003639 |
| | | | 18 | 0.053531 | 0.003055 |
| | | | 19 | 0.057380 | 0.011390 |
| | | | 20 | 0.061783 | 0.021945 |
| | | | 21 | 0.066927 | 0.036007 |
| | | | 22 | 0.073331 | 0.055932 |
| | | | 23 | 0.082153 | 0.088238 |
| | | | 24 | 0.100863 | 0.198122 |
| E(LU) | | | | 0.04643549 | |
| E(CP) | | | | -0.00984533 | |
| E(LB) | | | | | 0.02652435 |
| | 25 | 2 | 1 | -1.368683 | -0.494926 |
| | | | 2 | 2.368683 | 0.494926 |
| E(LU) | | | | 4.48592812 | |
| E(CP) | | | | 1.37873247 | |
| E(LB) | | | | | 0.49490321 |
| | 25 | 3 | 1 | -0.741297 | -0.328748 |
| | | | 2 | -0.706762 | -0.319677 |
| | | | 3 | 2.448059 | 0.648426 |
| E(LU) | | | | 2.08530608 | |
| E(CP) | | | | 0.74287178 | |
| E(LB) | | | | | 0.32648069 |
| | 25 | 4 | 1 | -0.468472 | -0.245175 |
| | | | 2 | -0.450043 | -0.241037 |
| | | | 3 | -0.415470 | -0.228750 |
| | | | 4 | 2.333986 | 0.714962 |
| E(LU) | | | | 1.18738033 | |
| E(CP) | | | | 0.46781327 | |
| E(LB) | | | | | 0.24222296 |
| | 25 | 5 | 1 | -0.321316 | -0.194777 |
| | | | 2 | -0.310282 | -0.193172 |
| | | | 3 | -0.288191 | -0.185159 |
| | | | 4 | -0.258993 | -0.173084 |
| | | | 5 | 2.178782 | 0.746191 |
| E(LU) | | | | 0.75602911 | |
| E(CP) | | | | 0.32008373 | |
| E(LB) | | | | | 0.19162842 |
| | 25 | 6 | 1 | -0.231589 | -0.161021 |
| | | | 2 | -0.224397 | -0.160862 |
| | | | 3 | -0.209240 | -0.155458 |
| | | | 4 | -0.188950 | -0.146734 |
| | | | 5 | -0.164473 | -0.135339 |
| | | | 6 | 2.018650 | 0.759414 |
| E(LU) | | | | 0.51744818 | |
| E(CP) | | | | 0.23032985 | |
| E(LB) | | | | | 0.15786311 |
| | 25 | 7 | 1 | -0.172349 | -0.136802 |
| | | | 2 | -0.167311 | -0.137524 |
| | | | 3 | -0.156342 | -0.133831 |
| | | | 4 | -0.141540 | -0.127351 |
| | | | 5 | -0.123602 | -0.118629 |
| | | | 6 | -0.102823 | -0.107905 |
| | | | 7 | 1.863966 | 0.762043 |
| E(LU) | | | | 0.37295445 | |
| E(CP) | | | | 0.17125667 | |
| E(LB) | | | | | 0.13371230 |
| | 25 | 8 | 1 | -0.130991 | -0.118557 |
| | | | 2 | -0.127221 | -0.119837 |
| | | | 3 | -0.118931 | -0.117327 |
| | | | 4 | -0.107714 | -0.112428 |
| | | | 5 | -0.094093 | -0.105611 |
| | | | 6 | -0.078289 | -0.097082 |
| | | | 7 | -0.060385 | -0.086925 |
| | | | 8 | 1.717625 | 0.757767 |
| E(LU) | | | | 0.27973288 | |
| E(CP) | | | | 0.13012998 | |
| E(LB) | | | | | 0.11556839 |

| N | M | I | A(N,M,I) | C(N,M,I) | |
|---|---|---|---|---|---|
| 25 | 9 | 1 | -0.100902 | -0.104300 | |
| | | 2 | -0.097898 | -0.105944 | |
| | | 3 | -0.091397 | -0.104281 | |
| | | 4 | -0.082624 | -0.100540 | |
| | | 5 | -0.071979 | -0.095134 | |
| | | 6 | -0.059629 | -0.088240 | |
| | | 7 | -0.045632 | -0.079936 | |
| | | 8 | -0.029985 | -0.070233 | |
| | | 9 | 1.5800[illegible]4 | 0.748609 | |
| E(LU) | | | 0.21673546 | | |
| E(CP) | | | | 0.10028244 | |
| E(LB) | | | | | 0.10142692 |
| 25 | 10 | 1 | -0.078304 | -0.092840 | |
| | | 2 | -0.075769 | -0.094722 | |
| | | 3 | -0.070501 | -0.093684 | |
| | | 4 | -0.063450 | -0.090817 | |
| | | 5 | -0.054925 | -0.086485 | |
| | | 6 | -0.045053 | -0.080848 | |
| | | 7 | -0.033874 | -0.073973 | |
| | | 8 | -0.021382 | -0.065870 | |
| | | 9 | -0.007535 | -0.056515 | |
| | | 10 | 1.450793 | 0.735755 | |
| E(LU) | | | 0.17263602 | | |
| E(CP) | | | | 0.07791785 | |
| E(LB) | | | | | 0.09008494 |
| 25 | 11 | 1 | -0.060903 | -0.083415 | |
| | | 2 | -0.058654 | -0.085451 | |
| | | 3 | -0.054256 | -0.084885 | |
| | | 4 | -0.048449 | -0.082692 | |
| | | 5 | -0.041474 | -0.079199 | |
| | | 6 | -0.033425 | -0.074550 | |
| | | 7 | -0.024332 | -0.068804 | |
| | | 8 | -0.014184 | -0.061972 | |
| | | 9 | -0.002945 | -0.054030 | |
| | | 10 | 0.009446 | -0.044927 | |
| | | 11 | 1.329176 | 0.719926 | |
| E(LU) | | | 0.14090683 | | |
| E(CP) | | | | 0.06073226 | |
| E(LB) | | | | | 0.08077665 |
| 25 | 12 | 1 | -0.047230 | -0.075517 | |
| | | 2 | -0.045150 | -0.077650 | |
| | | 3 | -0.041377 | -0.077446 | |
| | | 4 | -0.036489 | -0.075783 | |
| | | 5 | -0.030670 | -0.072958 | |
| | | 6 | -0.023991 | -0.069100 | |
| | | 7 | -0.016473 | -0.064265 | |
| | | 8 | -0.008104 | -0.058459 | |
| | | 9 | 0.001149 | -0.051664 | |
| | | 10 | 0.011340 | -0.043833 | |
| | | 11 | 0.022540 | -0.034894 | |
| | | 12 | 1.214454 | 0.701568 | |
| E(LU) | | | 0.11757719 | | |
| E(CP) | | | | 0.04725515 | |
| E(LB) | | | | | 0.07299117 |
| 25 | 13 | 1 | -0.036308 | -0.068792 | |
| | | 2 | -0.034320 | -0.070982 | |
| | | 3 | -0.031004 | -0.071058 | |
| | | 4 | -0.026804 | -0.069819 | |
| | | 5 | -0.021861 | -0.067534 | |
| | | 6 | -0.016228 | -0.064321 | |
| | | 7 | -0.009922 | -0.060231 | |
| | | 8 | -0.002924 | -0.055270 | |
| | | 9 | 0.004794 | -0.049420 | |
| | | 10 | 0.013273 | -0.042642 | |
| | | 11 | 0.022584 | -0.034867 | |
| | | 12 | 0.032798 | -0.026010 | |
| | | 13 | 1.105923 | 0.680946 | |
| E(LU) | | | 0.10012423 | | |
| E(CP) | | | | 0.03650889 | |
| E(LB) | | | | | 0.06637441 |
| 25 | 14 | 1 | -0.027463 | -0.062987 | |
| | | 2 | -0.025518 | -0.065206 | |
| | | 3 | -0.022537 | -0.065502 | |
| | | 4 | -0.018860 | -0.064606 | |
| | | 5 | -0.014588 | -0.062761 | |
| | | 6 | -0.009771 | -0.060083 | |
| | | 7 | -0.004406 | -0.056611 | |
| | | 8 | 0.001530 | -0.052347 | |
| | | 9 | 0.008022 | -0.047302 | |
| | | 10 | 0.015177 | -0.041393 | |
| | | 11 | 0.022982 | -0.034605 | |
| | | 12 | 0.031548 | -0.026830 | |
| | | 13 | 0.040941 | -0.017971 | |
| | | 14 | 1.002942 | 0.658203 | |
| E(LU) | | | 0.08688651 | | |
| E(CP) | | | | 0.02782135 | |
| E(LB) | | | | | 0.06067302 |
| 25 | 15 | 1 | -0.020217 | -0.057916 | |
| | | 2 | -0.018283 | -0.0601[illegible] | |
| | | 3 | -0.015551 | -0.06061[illegible] | |
| | | 4 | -0.012269 | -0.059993 | |
| | | 5 | -0.008537 | -0.058526 | |
| | | 6 | -0.004338 | -0.056280 | |
| | | 7 | 0.000300 | -0.053317 | |
| | | 8 | 0.005319 | -0.049695 | |
| | | 9 | 0.010966 | -0.045241 | |
| | | 10 | 0.016954 | -0.040149 | |
| | | 11 | 0.023615 | -0.034163 | |
| | | 12 | 0.030844 | -0.02732[illegible] | |
| | | 13 | 0.038774 | -0.019488 | |
| | | 14 | 0.047483 | -0.010542 | |
| | | 15 | 0.904940 | 0.633385 | |
| E(LU) | | | 0.07673648 | | |
| E(CP) | | | | 0.02071714 | |
| E(LB) | | | | | 0.05570065 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 25 | 16 | 1 | -0.014226 | -0.053437 |
| | | 2 | -0.012280 | -0.055655 |
| | | 3 | -0.009730 | -0.056262 |
| | | 4 | -0.006767 | -0.055881 |
| | | 5 | -0.003416 | -0.054698 |
| | | 6 | 0.000228 | -0.052868 |
| | | 7 | 0.004311 | -0.050319 |
| | | 8 | 0.008740 | -0.047138 |
| | | 9 | 0.013383 | -0.043435 |
| | | 10 | 0.018781 | -0.038783 |
| | | 11 | 0.024316 | -0.033639 |
| | | 12 | 0.030527 | -0.027559 |
| | | 13 | 0.037271 | -0.020612 |
| | | 14 | 0.044668 | -0.012645 |
| | | 15 | 0.052795 | -0.003537 |
| | | 16 | 0.811399 | 0.606468 |

E(LU) 0.06888924
E(CP) 0.01485185
E(LB) 0.05131672

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 25 | 17 | 1 | -0.009232 | -0.049443 |
| | | 2 | -0.007260 | -0.051640 |
| | | 3 | -0.004848 | -0.052357 |
| | | 4 | -0.002117 | -0.052162 |
| | | 5 | 0.000879 | -0.051264 |
| | | 6 | 0.004177 | -0.049710 |
| | | 7 | 0.007815 | -0.047517 |
| | | 8 | 0.011434 | -0.044984 |
| | | 9 | 0.015921 | -0.041405 |
| | | 10 | 0.020253 | -0.037606 |
| | | 11 | 0.025191 | -0.032939 |
| | | 12 | 0.030451 | -0.027620 |
| | | 13 | 0.036231 | -0.021444 |
| | | 14 | 0.042593 | -0.014304 |
| | | 15 | 0.049500 | -0.006173 |
| | | 16 | 0.057147 | 0.003248 |
| | | 17 | 0.721867 | 0.577320 |

E(LU) 0.06278626
E(CP) 0.00997094
E(LB) 0.04741317

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 25 | 18 | 1 | -0.005043 | -0.045848 |
| | | 2 | -0.003036 | -0.048015 |
| | | 3 | -0.000726 | -0.048819 |
| | | 4 | 0.001824 | -0.048779 |
| | | 5 | 0.004483 | -0.048170 |
| | | 6 | 0.007751 | -0.046642 |
| | | 7 | 0.010435 | -0.045269 |
| | | 8 | 0.014601 | -0.042266 |
| | | 9 | 0.017272 | -0.040246 |
| | | 10 | 0.022055 | -0.036060 |
| | | 11 | 0.026242 | -0.032037 |
| | | 12 | 0.030231 | -0.027809 |
| | | 13 | 0.035679 | -0.021917 |
| | | 14 | 0.041032 | -0.015644 |
| | | 15 | 0.047011 | -0.008309 |
| | | 16 | 0.053506 | 0.000123 |
| | | 17 | 0.060779 | 0.009946 |
| | | 18 | 0.635905 | 0.545761 |

E(LU) 0.05802293
E(CP) 0.00588284
E(LB) 0.04390460

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 25 | 19 | 1 | -0.001511 | -0.042582 |
| | | 2 | 0.000536 | -0.044712 |
| | | 3 | 0.002777 | -0.045580 |
| | | 4 | 0.005148 | -0.045706 |
| | | 5 | 0.007784 | -0.045118 |
| | | 6 | 0.010293 | -0.044292 |
| | | 7 | 0.013648 | -0.042297 |
| | | 8 | 0.016012 | -0.040961 |
| | | 9 | 0.019203 | -0.038460 |
| | | 10 | 0.024225 | -0.034053 |
| | | 11 | 0.026723 | -0.031592 |
| | | 12 | 0.030579 | -0.027487 |
| | | 13 | 0.034843 | -0.022690 |
| | | 14 | 0.040233 | -0.016383 |
| | | 15 | 0.045011 | -0.010159 |
| | | 16 | 0.050657 | -0.002512 |
| | | 17 | 0.056882 | 0.006342 |
| | | 18 | 0.063787 | 0.016770 |
| | | 19 | 0.553170 | 0.511471 |

E(LU) 0.05430050
E(CP) 0.00244102
E(LB) 0.04072222

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 25 | 20 | 1 | 0.001479 | -0.039587 |
| | | 2 | 0.003569 | -0.041673 |
| | | 3 | 0.005756 | -0.042596 |
| | | 4 | 0.008031 | -0.042816 |
| | | 5 | 0.010448 | -0.042449 |
| | | 6 | 0.013025 | -0.041554 |
| | | 7 | 0.015402 | -0.040540 |
| | | 8 | 0.018347 | -0.038621 |
| | | 9 | 0.021346 | -0.036313 |
| | | 10 | 0.024325 | -0.033953 |
| | | 11 | 0.028185 | -0.030127 |
| | | 12 | 0.030435 | -0.027632 |
| | | 13 | 0.035676 | -0.021855 |
| | | 14 | 0.036707 | -0.017912 |
| | | 15 | 0.043784 | -0.011387 |
| | | 16 | 0.048355 | -0.004818 |
| | | 17 | 0.053791 | 0.003245 |
| | | 18 | 0.059696 | 0.012672 |
| | | 19 | 0.066471 | 0.023804 |
| | | 20 | 0.473171 | 0.474112 |

E(LU) 0.05139[illegible]34
E(CP) -0.00046792
E(LB) 0.03780750

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 25 | 21 | 1 | 0.004015 | -0.036813 |
| | | 2 | 0.006150 | -0.038850 |
| | | 3 | 0.008301 | -0.039812 |
| | | 4 | 0.010494 | -0.040124 |
| | | 5 | 0.012767 | -0.039913 |
| | | 6 | 0.015083 | -0.039304 |
| | | 7 | 0.017547 | -0.038195 |
| | | 8 | 0.019926 | -0.036894 |
| | | 9 | 0.023362 | -0.034108 |
| | | 10 | 0.024924 | -0.033298 |
| | | 11 | 0.028334 | -0.029964 |
| | | 12 | 0.032751 | -0.025098 |
| | | 13 | 0.033983 | -0.023707 |
| | | 14 | 0.038804 | -0.017806 |
| | | 15 | 0.042477 | -0.012817 |
| | | 16 | 0.046778 | -0.006543 |
| | | 17 | 0.051285 | 0.000504 |
| | | 18 | 0.056394 | 0.009060 |
| | | 19 | 0.062243 | 0.019180 |
| | | 20 | 0.068693 | 0.031758 |
| | | 21 | 0.395689 | 0.432743 |
| E(LU) | | | 0.04914458 | |
| E(CP) | | | -0.00293163 | |
| E(LB) | | | | 0.03511308 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 25 | 22 | 1 | 0.006168 | -0.034211 |
| | | 2 | 0.008348 | -0.036193 |
| | | 3 | 0.010476 | -0.037183 |
| | | 4 | 0.012607 | -0.037570 |
| | | 5 | 0.014776 | -0.037485 |
| | | 6 | 0.017026 | -0.036955 |
| | | 7 | 0.019221 | -0.[illegible]36171 |
| | | 8 | 0.021340 | -0.035185 |
| | | 9 | 0.024679 | -0.032515 |
| | | 10 | 0.025895 | -0.032125 |
| | | 11 | 0.030229 | -0.027673 |
| | | 12 | 0.031117 | -0.027074 |
| | | 13 | 0.035583 | -0.0217[illegible] |
| | | 14 | 0.038063 | -0.018702 |
| | | 15 | 0.041779 | -0.013661 |
| | | 16 | 0.045324 | -0.008299 |
| | | 17 | 0.049433 | -0.001734 |
| | | 18 | 0.053748 | 0.005862 |
| | | 19 | 0.058821 | 0.015044 |
| | | 20 | 0.064336 | 0.026491 |
| | | 21 | 0.071016 | 0.040315 |
| | | 22 | 0.320015 | 0.386798 |
| E(LU) | | | 0.04741691 | |
| E(CP) | | | -0.00501984 | |
| E(LB) | | | | 0.03258909 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 25 | 23 | 1 | 0.007991 | -0.0[illegible]1733 |
| | | 2 | 0.010217 | -0.033654 |
| | | 3 | 0.012332 | -0.034661 |
| | | 4 | 0.014413 | -0.035114 |
| | | 5 | 0.016[illegible]07 | -0.035132 |
| | | 6 | 0.018615 | -0.034796 |
| | | 7 | 0.020791 | -0.034037 |
| | | 8 | 0.022797 | -0.03320[illegible] |
| | | 9 | 0.025487 | -0.031417 |
| | | 10 | 0.027226 | -0.030315 |
| | | 11 | 0.030451 | -0.027371 |
| | | 12 | 0.032439 | -0.025277 |
| | | 13 | 0.034975 | -0.022600 |
| | | 14 | 0.038427 | -0.018207 |
| | | 15 | 0.040862 | -0.014907 |
| | | 16 | 0.044536 | -0.009371 |
| | | 17 | 0.047835 | -0.003905 |
| | | 18 | 0.051722 | 0.003109 |
| | | 19 | 0.056077 | 0.011315 |
| | | 20 | 0.060795 | 0.021679 |
| | | 21 | 0.066[illegible]31 | 0.034261 |
| | | 22 | 0.073053 | 0.051189 |
| | | 23 | 0.245889 | 0.[illegible]34151 |
| E(LU) | | | 0.04611741 | |
| E(CP) | | | -0.00678580 | |
| E(LB) | | | | 0.03018924 |

| N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|
| 25 | 24 | 1 | 0.009526 | -0.029309 |
| | | 2 | 0.011794 | -0.031162 |
| | | 3 | 0.013905 | -0.032175 |
| | | 4 | 0.015953 | -0.032683 |
| | | 5 | 0.017988 | -0.032792 |
| | | 6 | 0.020022 | -0.032574 |
| | | 7 | 0.022074 | -0.032010 |
| | | 8 | 0.024017 | -0.031278 |
| | | 9 | 0.026504 | -0.029812 |
| | | 10 | 0.028155 | -0.028849 |
| | | 11 | 0.031096 | -0.026352 |
| | | 12 | 0.032808 | -0.024695 |
| | | 13 | 0.035503 | -0.021765 |
| | | 14 | 0.038100 | -0.018724 |
| | | 15 | 0.040722 | -0.015128 |
| | | 16 | 0.043691 | -0.010706 |
| | | 17 | 0.046790 | -0.005556 |
| | | 18 | 0.050125 | 0.000586 |
| | | 19 | 0.053932 | 0.007926 |
| | | 20 | 0.057987 | 0.017243 |
| | | 21 | 0.062967 | 0.028582 |
| | | 22 | 0.068493 | 0.043984 |
| | | 23 | 0.075661 | 0.065216 |
| | | 24 | 0.172188 | 0.272031 |
| E(LU) | | | 0.04517886 | |
| E(CP) | | | -0.00826855 | |
| E(LB) | | | | 0.02784672 |

TABLE OF WEIGHTS (CONTINUED)

| N | M | I | A(N,M,I) | C(N,M,I) | N | M | I | A(N,M,I) | C(N,M,I) |
|---|---|---|---|---|---|---|---|---|---|
| 25 | 25 | 1 | 0.010788 | -0.026796 | | | | | |
| | | 2 | 0.013098 | -0.028566 | | | | | |
| | | 3 | 0.015213 | -0.029573 | | | | | |
| | | 4 | 0.017239 | -0.030122 | | | | | |
| | | 5 | 0.019234 | -0.030313 | | | | | |
| | | 6 | 0.021211 | -0.030205 | | | | | |
| | | 7 | 0.023185 | -0.029800 | | | | | |
| | | 8 | 0.025057 | -0.029207 | | | | | |
| | | 9 | 0.027393 | -0.028041 | | | | | |
| | | 10 | 0.029004 | -0.027158 | | | | | |
| | | 11 | 0.031672 | -0.025206 | | | | | |
| | | 12 | 0.033363 | -0.023590 | | | | | |
| | | 13 | 0.035757 | -0.021260 | | | | | |
| | | 14 | 0.038198 | -0.018529 | | | | | |
| | | 15 | 0.040571 | -0.015429 | | | | | |
| | | 16 | 0.043268 | -0.011548 | | | | | |
| | | 17 | 0.046011 | -0.007108 | | | | | |
| | | 18 | 0.048983 | -0.001688 | | | | | |
| | | 19 | 0.052348 | 0.004773 | | | | | |
| | | 20 | 0.055864 | 0.013016 | | | | | |
| | | 21 | 0.060199 | 0.023073 | | | | | |
| | | 22 | 0.064907 | 0.036847 | | | | | |
| | | 23 | 0.071049 | 0.056035 | | | | | |
| | | 24 | 0.079353 | 0.087235 | | | | | |
| | | 25 | 0.097036 | 0.193159 | | | | | |
| E(LU) | | | -0.00009232 | | | | | | |
| E(CP) | | | | -0.00948561 | | | | | |
| E(LB) | | | | 0.02542407 | | | | | |

## APPENDIX D

## CRAMER-RAO EFFICIENCIES OF BEST LINEAR INVARIANT ESTIMATES OF PARAMETERS OF THE EXTREME-VALUE DISTRIBUTION

### INTRODUCTION AND SUMMARY

The Weibull distribution, since it was empirically derived by W. Weibull [12] in 1939, has been shown in innumerable situations to provide an appropriate model for survival populations associated with life testing. This distribution, which had been derived earlier by Von Mises and Fisher and Tippett (see Gumbel [3] ) using extreme-value considerations, is also known as the Fisher-Tippett type III distribution or the third asymptotic distribution of smallest values. This explains why the Weibull model is often applicable when the cause of failure is characterized by a weakest-link or severest-flaw phenomenon.

When there is no threshold below which a failure cannot occur, the following two-parameter form of the Weibull distribution may apply:

$$F_{\delta,b}(t) \begin{cases} = 1 - \exp[-(t/\delta)^{1/b}], & t \geq 0 \\ = 0 & \text{otherwise} . \end{cases}$$

Here $\delta > 0$ is a scale parameter and $b > 0$ determines the shape of the distribution. The location parameter present in the three-parameter form (when a lower failure threshold does exist) is assumed to be zero.

This two-parameter Weibull distribution can also be useful in providing a model for populations which are not of the survival type. A recent example is a study of the thrust build-up curves for a large rocket

engine. It was found that at any specified time from engine start the observed thrust levels for the specified engine system over many runs could be very well described by means of the two-parameter Weibull. It is not clear at this time why this distribution provides such a good model for the thrust build-up process.

If any random variable $T$ has the Weibull distribution with parameters $\delta$ and $b$, then $X = \log T$ has the (first) extreme-value distribution of smallest values, with

$$P(X \leq x) = F_{u,b}(x) = 1-\exp(-\exp((x-u)/b)), \quad b > 0.$$

The parameter $b$ is a scale parameter of this distribution and $u = \log \delta$ is the mode, a location parameter. For any estimation problem, an analysis in terms of $X$ rather than $T$ will be more fruitful.

One of the difficulties encountered in dealing with the two-parameter Weibull or extreme-value distribution has been the fact that "good" estimates of the parameters are not easily obtained. The maximum-likelihood estimates are obtained only with considerable effort (iteratively), and the mean squared errors of these estimates are not known except in the asymptotic case. Furthermore, since the minimally sufficient statistic for either the Weibull or extreme-value distribution consists of all the available extreme-value observations (see [10]), the dimension of the sufficient statistic is equal to the number of observations made.

Therefore, the result of Rao [11], which guarantees that maximum-likelihood estimates are the minimum-variance estimates of their expected values (when the number of sufficient statistics is equal to the number of parameters to be estimated), is not applicable when more than two observations are made. Hence one cannot claim optimality properties for the maximum-likelihood estimates of the two Weibull parameters when the number of observations is greater than two, but not large enough for asymptotic theory to apply.

A method in widespread use for Weibull parameter estimation employs a graphical plot which is based on fitting a linear model by approximating the method of least squares. The Weibull plot is of unquestionable value in finding outliers or in determining whether the sample data indicate that the population is of the prescribed type. With respect to the problem of estimation, however, graphical procedures which involve visually fitting the least-squares line will obviously be inferior to analytical procedures which yield true Gauss-Markov least-squares estimates.

The least-squares or best linear unbiased estimates which, for limited sample sizes, can be found relatively easily for certain functions of the two extreme-value parameters, once appropriate weights (or coefficients) have been found, enjoy all the usual asymptotic properties of consistency, efficiency, and normality provided by the maximum-likelihood

estimates (see Blom [1]). Furthermore, if the weights for obtaining these estimates are available, the covariance matrices of the estimates are available as well. A recent result given in [9] allows one to calculate as simple functions of the least-squares estimates and their covariance matrices, linear estimates which have smallest expected loss among all linear estimates with expected loss invariant under transformations of location and scale. Here loss is defined to be squared error divided by the square of the extreme-value scale parameter. These estimates are uniformly better than the best linear unbiased (BLU) estimates and are, in fact, the unique admissible minimax linear estimates. They, of course, also have the asymptotic properties mentioned above plus the property of asymptotic unbiasedness.

Calculation of the weights for obtaining the best linear unbiased estimates (based on the results of Lieblein [7]), and from these the best linear invariant estimates of the two extreme-value parameters, has been completed for censored samples up to size 25. These calculated weights, along with the variances and covariances of the estimates, appear in [10][1].

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[1] Although the calculations were performed for sample size $n$ ranging from 2 through 25 (an extension of tables of Lieblein [8] giving values for $n$ ranging from 2 through 6) the results given in [10] are through $n$ equal to 20 only.

In the present report expressions are derived for Cramér-Rao bounds for the expected squared error of regular invariant estimates of extreme-value location and scale parameters. The bounds are calculated and compared with the mean squared error of the best linear invariant estimates of the two extreme-value parameters and, in addition, similar comparisons are made for certain percentiles of the distribution, for censored samples up to size 25.

All of the estimation procedures considered above, along with all but the numerical results given below, are applicable to the general case in which any selection of observations is censored from a size $n$ sample. In the following, however, it is assumed that only the largest $n-r$, $2 \leq r \leq n$, observations are censored. This assumption corresponds to the situation in life testing in which testing is terminated at the time of the $r^{th}$ failure. Termination of a life test in this manner (by failure number) is called Type II censoring to distinguish it from the type of censoring (Type I) wherein testing is stopped at a specified time after the initiation of the test.

EFFICIENCIES OF BEST LINEAR INVARIANT ESTIMATES OF PARAMETERS OF THE EXTREME-VALUE DISTRIBUTION

Assume a population having the extreme-value distribution of smallest values given by

$$F_{u,b}(x) = 1-\exp(-\exp((x-u)/b)), \quad b > 0. \qquad (1)$$

The function $R_{u,b}(x) = 1-F_{u,b}(x) = \exp(-\exp((x-u)/b))$ gives the proportion of the population lying to the right of the point $x$. Assume a specified value $R$ for $R_{u,b}(x)$ and let $x_R$ be the point defined by $R = \exp(-\exp((x_R-u)/b))$. Then $x_R = u+b \log\log(1/R)$ is the percentile above which 100R% of the population lies. If $R = 1/e$, then $x_R = u$. In a life-testing situation $R(x)$ is the proportion of the population surviving at log time $x$, so that if a reliability or survival proportion $R$ is specified, the corresponding $x_R$ is the logarithm of the time at which 100R% of the population will have survived.

Let a size $n$ sample be chosen from the population with distribution (1) and let an estimate of $x_R$ be based on $X_{(1,n)} \leq X_{(2,n)} \ldots \leq X_{(r,n)}$, the first $r$, $2 \leq r \leq n$, of the $n$ ordered observations. In [10] there is considered the class of linear estimates of $u$, $b$, and $x_R$ with expected loss invariant under location and scalar transformations, where loss is defined as squared error divided by $b^2$. It is shown that

the best (minimum-expected-loss) estimates in this class are linear functions of the BLU estimates of $u$ and $b$. Let $u^*$ with variance $Ab^2$ and $b^*$ with variance $Cb^2$ be the BLU estimates of $u$ and $b$, respectively, and let $Bb^2$ be the covariance of the two estimates. The BLU estimate of $x_R = u+b \log\log(1/R)$ is $X_R^* = u^* + b^* \log\log(1/R)$, with variance $Q_R b^2 = [A+2B \log\log(1/R) + C(\log\log(1/R))^2]b^2$. The best linear invariant, BLI, estimate of $x_R$ is $\tilde{X}_R = u^* - b^*[B/(1+C)] + [b^*/(1+C)]\log\log(1/R) \equiv \tilde{u} + \tilde{b}\log\log(1/R)$. The expected losses for $\tilde{u}$ and $\tilde{b}$ are $\tilde{A}b^2 = [A-B^2/(1+C)]b^2$ and $\tilde{C}b^2 = [C/(1+C)]b^2$, respectively, with $E(\tilde{u}\tilde{b} - ub) \equiv \tilde{B}b^2 = [B/(1+C)]b^2$. For $\tilde{X}_R$, the expected loss is thus $\tilde{Q}_R b^2 = [\tilde{A} + 2\tilde{B}\log\log(1/R) + \tilde{C}(\log\log(1/R))^2]b^2$ or $\tilde{Q}_R b^2 = \{A + 2B\log\log(1/R) + C(\log\log(1/R))^2 - [(B+C\log\log(1/R))^2/(1+C)]\}b^2$, clearly uniformly less than $Q_R b^2$. One may then inquire as to the efficiency of this estimate with respect to any nonlinear estimate which could be obtained.

Since the expected value of $\tilde{X}_R$ is equal to $u+b\log\log(1/R) - [(B+C\log\log(1/R))/(1+C)]b$, this estimate is, in general, biased. Cramér-Rao bounds for the variance of unbiased estimates, therefore constitute for each $r$ and $n$ an incorrect standard with which to compare the mean squared error (MSE) of estimates with this given bias.

The variance $nQ_R^0 b^2 = n\,[A^0 + 2B^0 \log\log(1/R) + C^0(\log\log(1/R))^2]b^2$ of an efficient estimate of $\sqrt{n}x_R$ based on $X_{(1,n)}, X_{(2,n)}, \ldots, X_{(r,n)}$ depends upon $r$ and $n$ through $p = r/n$ only. The expression giving the appropriate lower bounds for the mean squared error of biased estimates of $x_R$ as a function of $A^0$, $B^0$, and $C^0$ and any given bias is derived in Appendix D.2.

This expression for estimates of $x_R$ with bias $-[(B+C\log\log(1/R))/(1+C)]b$ is

$$[A^0 - \frac{2B(B^0)(1+C)-B^2(1+C^0)}{(1+C)^2} +$$

$$\frac{2[B^0(1+C)+B(C-C^0)]\log\log(1/R)}{(1+C)^2} + \frac{C^0+C^2}{(1+C)^2}(\log\log(1/R))^2]b^2$$

$$\equiv [\tilde{A}^0 + 2\tilde{B}^0 \log\log(1/R) + \tilde{C}^0(\log\log(1/R))^2]b^2 .$$

This preliminary result is not, however, particularly meaningful as a lower bound, since other good invariant estimates which are nonlinear and have bias different from that specified may exist. A more meaningful lower bound would be one for the MSE of invariant estimates of $x_R$, a natural analogue to the classical bound $Q_R^0$ which provides the standard with which one may compare the variances of unbiased estimates of this parameter. Expressions giving the lower bounds for the MSE of invariant estimates of general location and scale parameters are therefore derived in Appendix D.1 and applied to the location parameter $x_R$

(see footnote, p. 149) and the scale parameter b. These bounds for $x_R$ and b are

$$\tilde{Q}_R^0 b^2 = \left\{A^0 + 2B^0 \log\log(1/R) + C^0(\log\log(1/R))^2 - (B^0 + C^0\log\log(1/R))^2/(1+C^0)\right\} b^2$$

$$= \left\{\tilde{A}^0 + 2\tilde{B}^0 \log\log(1/R) + \tilde{C}^0(\log\log(1/R))^2\right\} b^2$$

and $$\left[C^0/(1+C^0)\right] b^2 = \tilde{C}^0 b^2 ,$$

respectively.

Table D.I gives values for $\tilde{Q}_R$ and $\tilde{Q}_R^0$, with $R = .90$ and $R = .95$, and $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{A}^0$, $\tilde{B}^0$, and $\tilde{C}^0$ for $2 \le n \le 25$, $2 \le r \le n$. Table D.II gives values for A, B, C, $A^0$, $B^0$, $C^0$, $Q_R$, and $Q_R^0$ for the same values of R, r, and n so comparisons can be made among the various sets of expected losses. The values for A, B, and C are obtained from [10] as are the expressions for $A^0$, $B^0$, and $C^0$. The Cramér-Rao efficiency of $\tilde{X}_R$ for any combination of R, r, and n (and with b considered a nuisance parameter) is simply $\tilde{Q}_R^0$ divided by $\tilde{Q}_R$. Cramér-Rao efficiencies for $\tilde{u}$ ($\tilde{X}_R$ with $R = 1/e$ and b a nuisance parameter) will be given by $\tilde{A}^0/\tilde{A}$. Efficiencies for $\tilde{b}$ with $x_R$ a nuisance parameter will be given by $\tilde{C}^0/\tilde{C}$. The values of the efficiencies of $\tilde{Q}_{.90}$ for $2 \le n \le 18$, $2 \le r \le n$ appear in Table D.III.

It can be seen from Table D.III that for $n \leq 18$ the Cramér-Rao efficiencies of $\tilde{Q}_{.90}$ are greater than .86 when $r$ is greater than 2. When $r$ is equal to 2 $(X_{(1,n)}, X_{(2,n)})$ is a complete sufficient statistic for $x_R$ and $b$, as is shown in [10]. The estimates $X_R^*$ and $b^*$, therefore, are the unique (with probability 1) uniformly minimum-variance unbiased estimates. It can be further inferred on the basis of the results of Appendix D.1 that, for $r = 2$, $\tilde{X}_R$ and $\tilde{b}$ are best in the class of all invariant estimates of $x_R$ and $b$ respectively, so that all the bounds given in Table D.I, as well as in Table D.II, are unrealistically low for $r$ equal to 2 at least.

From Table D.I it can be observed that, for $r \geq 8$, all of the values given for expected loss indicate that the corresponding estimates have efficiencies of at least .84 with respect to their lower bounds.

A comparison of Table D.I and Table D.II shows that the discrepancy between the MSE of the BLI estimates and that of the BLU estimates is most pronounced when $n$ is small and/or when there is extensive censoring and when a specified $R$ is large.

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## APPENDIX D.1

### CRAMÉR-RAO LOWER BOUNDS FOR INVARIANT ESTIMATES OF LOCATION AND SCALE PARAMETERS

Let $Y_1, Y_2, \ldots, Y_r$ be the first $r$ order statistics of a size $n$ random sample chosen from a population with density $f_{\mu,\sigma}(x)$, where $(\mu,\sigma)$ is a location-scale parameter[1] with $\sigma > 0$. Let the joint density of $Y_1, Y_2, \ldots, Y_r$ be $h_{\mu,\sigma}(y_1, y_2, \ldots, y_r)$, let $p = r/n$, and let $\hat{\mu}$ and $\hat{\sigma}$ be the maximum-likelihood estimates, based on $Y_1, Y_2, \ldots, Y_r$, of $\mu$ and $\sigma$, respectively.

It is assumed that the regularity conditions given in [4] are satisfied so that the concentration ellipse of the joint distribution of any pair of regular unbiased estimates of $\sqrt{n}\mu$ and $\sqrt{n}\sigma$ (where the definition of regular estimation is given in Appendix B with $\underset{\sim}{\theta} = (\mu,\sigma)$) contains the fixed ellipse specified by the asymptotic variances and covariances of $\sqrt{n}\hat{\mu}$ and $\sqrt{n}\hat{\sigma}$. Thus, the variance of any regular unbiased estimate

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[1] Here, if a density $f(x|\underset{\sim}{\theta})$ with respect to Lebesgue measure exists, $\underset{\sim}{\theta} = (\theta_1,\theta_2)$ is a location-scale parameter if and only if $f(x|\underset{\sim}{\theta}) = \frac{1}{\theta_2} g\left(\frac{x-\theta_1}{\theta_2}\right)$ for some function $g$. If $f(x|\underset{\sim}{\theta})$ exists and $(\theta_1,\theta_2)$ is a location-scale parameter, then for $c$ a known constant, $(\theta_1+c\theta_2, c\theta_2) \equiv (\theta_1', \theta_2')$ is a location-scale parameter, since $f(x|\theta_1',\theta_2') = \frac{c}{\theta_2'} g\left(\frac{x-\theta_1'+\theta_2'}{\theta_2'/c}\right) = \frac{1}{\theta_2'} \eta\left(\frac{x-\theta_1'}{\theta_2'}\right)$ for some $\eta$.

of $\sqrt{n}\mu$, where $\sigma$ is considered a nuisance parameter, is greater than or equal to $n\alpha^0\sigma^2$, the asymptotic variance of $\sqrt{n}\hat{\mu}$, and likewise, the variance of any regular unbiased estimate of $\sqrt{n}\sigma$ is greater than or equal to $n\gamma^0\sigma^2$, the asymptotic variance of $\sqrt{n}\hat{\sigma}$ (see Cramér). The variances $n\alpha^0\sigma^2$ and $n\gamma^0\sigma^2$ depend upon $r$ and $n$ through $p = r/n$ only, but $\alpha^0$ and $\gamma^0$ are functions of both $r$ and $n$.

Let loss be defined as squared error divided by $\sigma^2$ and consider estimates of $\varphi = \ell\mu + m\sigma$ ($\ell$ and $m$ known constants) with expected loss invariant under transformations of location and scale. Such an estimate has expected loss (risk) independent of $\mu$ and $\sigma$ for all $\mu$ and $\sigma$. It is desired to find an expression giving a lower bound for the risk of any regular invariant estimate $\bar{\varphi}$ of $\varphi$ when $\ell = 1$, $m = 0$ or $m = 1$, $\ell = 0$.

The requirement of invariant risk prescribes a particular form for $\bar{\varphi}$. Let $R(\varphi,\bar{\varphi})$ represent the expected loss of $\bar{\varphi}$ in the general case in which $\varphi = \ell\mu + m\sigma$. $R(\varphi,\bar{\varphi})$ is independent of $\mu$ and $\sigma$ for all values of $\mu$ and $\sigma$ and is equal to

$$V(\bar{\varphi})/\sigma^2 + [E(\bar{\varphi}-\varphi)]^2/\sigma^2 ,$$

where $V(\bar{\varphi})$ is the variance of $\bar{\varphi}$ and $E(\bar{\varphi}-\varphi)$ is the bias of this estimate. Both terms are greater than or equal to zero so that each must be independent of $\mu$ and $\sigma$ for all $\underset{\sim}{\theta} = (\mu,\sigma)$. Thus, since

$E(\bar{\varphi}-\varphi)$ is equal to $E[\bar{\varphi}-\ell\mu-m\sigma]$, $\bar{\varphi}$ must be equal to $\ell\mu^* + k\sigma^*$, where $\mu^*$ and $\sigma^*$ are regular unbiased estimates of $\mu$ and $\sigma$, respectively, and k is independent of $\mu$ and $\sigma$ for all $\underset{\sim}{\theta}$. The risk of $\bar{\varphi}$ is therefore of the form

$$\frac{1}{\sigma^2}\Big\{\ell^2V(\mu^*) + 2\ell kC(\mu^*,\sigma^*) + k^2V(\sigma^*)$$
$$+ \Big[E(\ell\mu^* + k\sigma^* - \ell\mu - m\sigma)\Big]^2\Big\},$$

where $V(\mu^*)$ is the variance of $\mu^*$, $V(\sigma^*)$ is the variance of $\sigma^*$, and $C(\mu^*,\sigma^*)$ is the covariance of $\mu^*$ and $\sigma^*$. Furthermore,

$$\frac{1}{\sigma^2}V(\bar{\varphi}) = \frac{1}{\sigma^2}\left[\ell^2V(\mu^*) + 2\ell kC(\mu^*,\sigma^*) + k^2V(\sigma^*)\right]$$

is independent of $\mu$ and $\sigma$ for all $\underset{\sim}{\theta}$. Therefore, $V(\mu^*)$, $C(\mu^*\sigma^*)$, and $V(\sigma^*)$ are of the form $\alpha\sigma^2$, $\beta\sigma^2$, and $\gamma\sigma^2$, respectively, with $\alpha$ and $\gamma$ positive, and for all $\mu$ and $\sigma$, either (1) $\alpha$, $\beta$, and $\gamma$ are all independent of $\mu$ and $\sigma$, (2) $\alpha$ is independent of $\mu$ and $\sigma$ and $2\ell k\beta(\mu,\sigma) \equiv -k^2\gamma(\mu,\sigma)$, (3) $\gamma$ is independent of $\mu$ and $\sigma$ and $2\ell k\beta(\mu,\sigma) \equiv -\ell^2\alpha(\mu,\sigma)$ or (4) $\beta$ is independent of $\mu$ and $\sigma$ and $\ell^2\alpha(\mu,\sigma) \equiv -k^2\gamma(\mu,\sigma)$. The estimate $\bar{\varphi}$ cannot, of course, be invariant if $V(\bar{\varphi}) \equiv 0$ unless $\bar{\varphi} = \ell\mu + k\sigma$ for all $\mu$ and $\sigma$.

One may now make use of the fact that the correlation coefficient of $\mu^*$ and $\sigma^*$ is less than or equal to 1, which implies that $\beta^2 \leq \alpha\gamma$

and hence that $V(\tilde{\varphi})/\sigma^2 \leq \ell^2\alpha + |2\ell k\sqrt{\alpha\gamma}| + k^2\gamma$. Then, since $V(\tilde{\varphi})/\sigma^2$, $\ell$, and $k$ are independent of $\mu$ and $\sigma$ for all $\underset{\sim}{\theta}$, if $\alpha$ is a constant, so is $\gamma$. Conversely, if $\gamma$ is a constant, so is $\alpha$. Also, $\ell^2\alpha > 0$ cannot be identically equal to $-k^2\gamma < 0$. Therefore $\alpha$, $\beta$, and $\gamma$ are all independent of $\mu$ and $\sigma$ for all values of the parameters.

If $R(\varphi,\bar{\varphi})$ is then minimized with respect to $k$, and the $k$ which yields the minimum value for $R(\varphi,\bar{\varphi})$ is substituted into the expression for $\bar{\varphi}$, the resulting expression defines the best estimate $\tilde{\varphi}$ of $\varphi$ among invariant estimates based on a given $\mu^*$ and $\sigma^*$. Since the minimizing $k$ is equal to $(m-\ell\beta)/(1+\gamma)$, $\tilde{\varphi}$ is equal to $\ell\mu^*+[(m-\ell\beta)/(1+\gamma)]\sigma^*$ and $R(\varphi,\tilde{\varphi})$, the expected loss of $\tilde{\varphi}$, is equal to
$\ell^2\alpha + 2\ell m\beta + m^2\gamma - (m\gamma + \ell\beta)^2/(1+\gamma)$.

The change in $R(\varphi,\tilde{\varphi})$ induced by changes in $\alpha$, $\beta$, and $\gamma$, corresponding to changes in $\mu^*$ and $\sigma^*$, is now considered. Since $R(\varphi,\tilde{\varphi})$ cannot be negative, it is clearly smallest when $\alpha$, $\beta$, and $\gamma$ are all zero. Moreover, $\partial R(\varphi,\tilde{\varphi})/\partial\alpha$ is equal to $\ell^2 \geq 0$ and $\partial R(\varphi,\tilde{\varphi})/\partial\gamma$ is equal to $(m-\ell\beta)^2/(1+\gamma)^2 \geq 0$ so that $R(\varphi,\tilde{\varphi})$ is monotonically non-decreasing in both $\alpha$ and $\gamma$ for fixed $\beta$. It can be shown, too, that for $m = 1$, $\ell = 0$ and for $\ell = 1$, $m = 0$, an increase in both $\alpha$ and $\gamma$ will effect an increase in $R(\varphi,\tilde{\varphi})$, irrespective of any change in $\beta$.

Let $m = 1$, $\ell = 0$ so that $\varphi = \sigma$ and $\tilde{\varphi} = \tilde{\sigma}$. Then, since $\partial R(\varphi,\tilde{\varphi})/\partial\beta = 2\ell(m-\ell\beta)/(1+\gamma)$, $R(\sigma,\tilde{\sigma})$ is monotonically increasing in $\gamma$, independently of changes in $\beta$. $R(\sigma,\tilde{\sigma})$ is also independent of $\alpha$. Let $m = 0$, $\ell = 1$ so that $\varphi = \mu$ and $\tilde{\varphi} = \tilde{\mu}$. One may now again make use of the fact that $\beta^2 \le \alpha\gamma$, which implies that $d\beta \le |(\gamma/2\beta)d\alpha|$ and hence that $|\frac{\partial R}{\partial\beta}\, d\beta| = |[2\beta/(1+\gamma)]d\beta| \le [\gamma/(1+\gamma)]\,|d\alpha|$. Then,

$dR(\mu,\tilde{\mu}) \ge d\alpha - [\gamma/(1+\gamma)]\,|d\alpha| + [\beta^2/(1+\gamma)^2]d\gamma$ or

$dR(\mu,\tilde{\mu}) \ge [1-\gamma/(1+\gamma)]d\alpha + [\beta^2/(1+\gamma)^2]d\gamma > 0$ for $d\alpha > 0$ and $d\gamma > 0$.

Thus, an increase in both $\alpha$ and $\gamma$ induces an increase in $R(\mu,\tilde{\mu})$. Therefore, the expected squared error of $\tilde{\mu}$, any regular invariant estimate of $\mu$, is greater than or equal to $[\alpha^0-\beta^{0^2}/(1+\gamma^0)]\sigma^2$ (where $n\beta^0$ is the asymptotic covariance of $\sqrt{n}\hat{\mu}$ and $\sqrt{n}\hat{\sigma}$), and the mean squared deviation of $\tilde{\sigma}$, any regular invariant estimate of $\sigma$, is greater than or equal to $[\gamma^0/(1+\gamma^0)]\sigma^2$, since $\gamma \ge \gamma^0$ and $\alpha \ge \alpha^0$.

In order to apply this result to the parameters of the extreme-value distribution, we let $\mu$ be the location parameter $x_R$ (see footnote, p. 12) and $\sigma$ be the scale parameter $b$. All regularity conditions are satisfied, and the asymptotic variance of the maximum-likelihood estimate of $\sqrt{n}x_R$ (corresponding to $n\alpha^0\sigma^2$) is $n[A^0+2B^0\log\log(1/R) + C^0(\log\log(1/R))^2]b^2$. Likewise, $n[B^0+C^0\log\log(1/R)]b^2$ corresponds to $n\beta^0\sigma^2$ and $nC^0b^2$ corresponds to $n\gamma^0\sigma^2$. Therefore, the expected squared deviations of any regular

invariant estimates of $x_R$ and of $b$ are less than or equal to

$$\tilde{Q}_R^0 b^2 = \Big\{A^0 + 2B^0 \log\log(1/R) + C^0(\log\log(1/R))^2$$

$$- (B^0 + C^0 \log\log(1/R))^2/(1+C^0)\Big\}\, b^2$$

$$\equiv \Big\{\tilde{A}^0 + 2\tilde{B}^0 \log\log(1/R) + \tilde{C}^0(\log\log(1/R))^2\Big\}\, b^2$$

and $[C^0/(1+C^0)]b^2 \equiv \tilde{C}^0 b^2$, respectively.

## APPENDIX D.2

## CRAMÉR-RAO LOWER BOUNDS FOR ESTIMATES WITH GIVEN BIAS WHEN A NUISANCE PARAMETER IS PRESENT AND THE SAMPLE IS CENSORED

Let $Y_1, Y_2, \ldots, Y_r$ be the first $r$ order statistics of a size $n$ random sample chosen from a population with density $f_{\theta}(x)$, where $\theta = (\theta_1, \theta_2)$, and let the joint density of $Y_1, Y_2, \ldots, Y_r$ be given by $h_{\underset{\sim}{\theta}}(y_1, y_2, \ldots, y_r)$. Suppose that all regularity conditions for $f_{\underset{\sim}{\theta}}(x)$ are satisfied (see Halperin [4] ) so that the maximum-likelihood estimates $\hat{\theta}_1$ and $\hat{\theta}_2$ of $\theta_1$ and $\theta_2$, respectively, based on $Y_1$, $Y_2, \ldots, Y_r$, converge in probability to the true values as $n \longrightarrow \infty$ with $p = r/n$ fixed. Moreover, the concentration ellipse of the joint asymptotic distribution of $\hat{\theta}_1$ and $\hat{\theta}_2$ lies wholly within the concentration ellipse of any pair of regular unbiased estimates of $\theta_1$ and $\theta_2$. Under the same conditions, the asymptotic covariance matrix of the maximum-likelihood estimates of $\sqrt{n}\theta_1$ and $\sqrt{n}\theta_2$ is $\underset{\sim}{D}^{-1}$ where

$$\underset{\sim}{D} = \left\{d_{i,j}\right\} = \lim_{\substack{n \longrightarrow \infty \\ p \text{ fixed}}} \left\{\lambda_{i,j}/n\right\} ,$$

and
$$\lambda_{i,j} = -E\left[\frac{\partial^2 \log h_{\underset{\sim}{\theta}}(y_1, y_2, \ldots, y_r)}{\partial\theta_i \partial\theta_j}\right] .$$

Regular estimation of $\theta_1$ and $\theta_2$ is defined as in Halperin [4] and Cramér [2]. It is supposed that $Y_1, Y_2, \ldots, Y_r$ can be transformed to new variables $\lambda_1, \lambda_2, \ldots, \lambda_{r-2}, \bar{\theta}_1, \bar{\theta}_2$ such that

$$h_{\underset{\sim}{\theta}}(y_1, y_2, \ldots, y_r) \prod_{i=1}^{r} dy_i$$

$$= g_{\underset{\sim}{\theta}}(\bar{\theta}_1, \bar{\theta}_2) q_{\underset{\sim}{\theta}}(\lambda_1, \lambda_2, \ldots, \lambda_{r-2} | \bar{\theta}_1, \bar{\theta}_2) \prod_{i=1}^{r-2} d\lambda_i d\bar{\theta}_1 d\bar{\theta}_2 ,$$

where $g_{\underset{\sim}{\theta}}(\bar{\theta}_1, \bar{\theta}_2)$ is the joint density function of $\bar{\theta}_1$ and $\bar{\theta}_2$ and $q_{\underset{\sim}{\theta}}(\lambda_1, \lambda_2, \ldots, \lambda_{r-2} | \bar{\theta}_1, \bar{\theta}_2)$ is the conditional density of $\lambda_1, \lambda_2, \ldots, \lambda_{r-2}$ given $\bar{\theta}_1$ and $\bar{\theta}_2$. If $\partial h/\partial\theta_j$, $\partial q/\partial\theta_j$, and $\partial g/\partial\theta_j$ exist for every $\theta_j$ and if

$$\frac{\partial h}{\partial\theta_j} < H_0(y_1, y_2, \ldots, y_r),\ \frac{\partial g}{\partial\theta_j} < G_0(\bar{\theta}_1, \bar{\theta}_2),\ \frac{\partial q}{\partial\theta_j} < M_0(\lambda_1, \lambda_2, \ldots, \lambda_{r-2}, \bar{\theta}_1, \bar{\theta}_2),$$

$j = 1,2$, where $H_0$; $\bar{\theta}_1 G_0$, $\bar{\theta}_2 G_0$, and $G_0$; and $M_0$ are integrable over the whole space of $(y_1, y_2, \ldots, y_r)$; $\bar{\theta}_1, \bar{\theta}_2$; and $\bar{\theta}_1, \bar{\theta}_2, \lambda_1, \lambda_2, \ldots, \lambda_{r-2}$, respectively, then $\bar{\theta}_1$ and $\bar{\theta}_2$ will be called regular joint estimates of $\theta_1$ and $\theta_2$.

The variance of any regular unbiased estimate of $\theta_1$ is greater than or equal to $\frac{1}{n} d_{1,1}^{-1}$, the asymptotic variance of $\sqrt{n}\hat{\theta}_1$ (see Cramér, p. 494). Furthermore, since a reparameterization of $f_{\underset{\sim}{\theta}}(x)$ may give $\underset{\sim}{\theta} = (\eta(\theta_1, \theta_2), \theta_2)$, the variance of any regular unbiased estimate of

$\eta(\theta_1,\theta_2)$, $\eta$ nonsingular, is less than that of $\sqrt{n}\,\eta(\hat\theta_1,\hat\theta_2)$ for $n\to\infty$.

It can now be shown that the following is true of the mean squared error (MSE) of $\bar\theta_1$, any regular estimate of $\theta_1$. If $\bar\theta_1$ is a regular estimate of $\theta_1$ and the bias of $\bar\theta_1$ is $k_1\theta_1$, with $k_1$ independent of $\underset{\sim}{\theta}$, then MSE $(\bar\theta_1) \geq \frac{1}{n}(1+k_1)^2 d^{-1}_{1,1} + (k_1\theta_1)^2$.

If the bias of $\theta_1$ is $k_2\theta_2$, with $k_2$ independent of $\underset{\sim}{\theta}$, then $\text{MSE}(\bar\theta_1) \geq \frac{1}{n}(d^{-1}_{1,1}+2k_2d^{-1}_{1,2}+k_2^2d^{-1}_{2,2})+(k_2\theta_2)^2$.

Proof:

If $\bar\theta_1$ has bias $k_1\theta_1$, then $E(\bar\theta_1) = \theta_1+k_1\theta_1 = (1+k_1)\theta_1$. If a reparameterization is made so that $\underset{\sim}{\theta} = ((1+k_1)\theta_1,\theta_2)$, then, since $h_{\theta_1,\theta_2}(y_1,y_2,\dots,y_r) = h_{(1+k_1)\theta_1,\theta_2}(y_1,y_2,\dots,y_r)$, $\sqrt{n}\bar\theta_1$ is by definition a regular unbiased estimate of $\sqrt{n}(1+k_1)\theta_1$ with variance greater than or equal to the variance of the maximum-likelihood estimate of $\sqrt{n}(1+k_1)\theta_1$ as $n\to\infty$, p fixed.

If $\bar\theta_1$ has bias $k_2\theta_2$, then $E(\bar\theta_1) = \theta_1+k_2\theta_2$ and $\sqrt{n}\bar\theta_1$ is a regular unbiased estimate of $\sqrt{n}(\theta_1+k_2\theta_2)$ with variance greater than or equal to the maximum-likelihood estimate of this parametric function of $\theta_1$ and $\theta_2$ as $n\to\infty$, p fixed. Then since the maximum-likelihood estimate of $c_1\theta_1 + c_2\theta_2$ is $c_1\hat\theta_1 + c_2\hat\theta_2$, with variance $c_1^2\text{Var}(\hat\theta_1)+2c_1c_2\text{Cov}(\hat\theta_1,\hat\theta_2) + c_2^2\text{Var}(\hat\theta_2)$, $\text{MSE}(\bar\theta_1) \geq \frac{1}{n}(1+k_1)^2d^{-1}_{1,1}+ (k_1\theta_1)^2$

when $E(\bar{\theta}_1) = (1+k_1)\theta_1$, and $MSE(\bar{\theta}_1) \geq \frac{1}{n}(d^{-1}_{1,1}+2k_2d^{-1}_{1,2}+k_2^2d^{-1}_{2,2})+(k_2\theta_2)^2$ when $E(\bar{\theta}_1) = \theta_1+k_2\theta_2$.

It should be noted that this result agrees with one concerning more general forms of bias, which could be fairly directly derived for the uncensored case on the basis of a lemma of Hodges appearing in [6]. If this derivation is made for the case of a single unknown parameter ($\underset{\sim}{\theta} = \theta_1$, with $E(\bar{\theta}_1) = \theta$ + bias as it is in [5] for a Gaussian distribution with unknown $\sigma$, the bound obtained is larger than that given by Cramér in [2] for the single-parameter case.

For the problem considered in this paper all regularity conditions are satisfied. The parameter point $\underset{\sim}{\theta} = (\theta_1, \theta_2)$ is given by $\theta_1 = x_R = u + b \log\log(1/R)$ and $\theta_2 = b$, and the prescribed bias is $[-(B+C \log\log(1/R)/(1+C)]\, b \equiv k_2 b$, with $k_2$ a function of R, r, and n.

The covariance matrix $\frac{1}{n}\underset{\sim}{D}^{-1}$ for joint efficient estimates of $x_R$ and b is

$$\begin{bmatrix} [A^0 + 2B^0\log\log(1/R) + C^0\log^2\log(1/R)]b^2 & [B^0 + C^0\log\log(1/R)]b^2 \\ [B^0 + C^0\log\log(1/R)]b^2 & C^0b^2 \end{bmatrix}$$

Thus, the Cramér-Rao lower bound for mean squared error of estimates

of $x_R$ with bias $k_2 b$, is

$$\left\{A^0 - \frac{2BB^0(1+C)-B^2(1+C^0)}{(1+C)^2} + \right.$$

$$\left. \frac{2[B^0(1+C)+B(C-C^0)]\log\log(1/R)}{(1+C)^2} + \frac{(C^0+C^2)(\log\log(1/R))^2}{(1+C)^2}\right\}b^2$$

Likewise, the corresponding bound for estimates of $b(\theta_1=b,\ \theta_2=x_R)$ with bias $[-C/(1+C)]b \equiv k_1 b$ is $[(C^0+C^2)/(1+C)^2]b^2$.

Table D.I - Factors for Calculating Cramér-Rao Efficiencies of BLI Estimates of a Scale Parameter and 100(1-R) Percent Points of the Extreme-Value Distribution, $R = e^{-1}$, .90, .99, Where $\frac{n-r}{n}$ of the Size n Ordered Sample is Censored From Above, $2 \le n \le 25$, $2 \le r \le n$

| n | r | $\tilde{A}$ | $\tilde{A}^0$ | $\tilde{B}$ | $\tilde{B}^0$ | $\tilde{C}$ | $\tilde{C}^0$ | $\tilde{Q}_{.90}$ | $\tilde{Q}^0_{.90}$ | $\tilde{Q}_{.99}$ | $\tilde{Q}^0_{.99}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 0.657130 | 0.541667 | 0.037574 | -0.098554 | 0.415839 | 0.233107 | 2.593892 | 2.165725 | [illegible] | 6.381769 |
| 3 | 2 | 0.795461 | 0.510573 | 0.257510 | 0.054933 | 0.450055 | 0.285418 | 1.915628 | 1.703733 | 7.950090 | 6.065004 |
|  | 3 | 0.402407 | 0.363452 | -0.018422 | -0.071238 | 0.256346 | 0.168498 | 1.783495 | 1.537373 | 5.996531 | 4.584506 |
| 4 | 2 | 1.014778 | 0.589262 | 0.413509 | 0.163705 | 0.464388 | 0.300232 | 1.505415 | 1.372892 | 7.037457 | 5.436456 |
|  | 3 | 0.423151 | 0.333607 | 0.084776 | 0.007460 | 0.281729 | 0.203175 | 1.468320 | 1.328941 | 5.604971 | 4.564433 |
|  | 4 | 0.292477 | 0.273582 | -0.028312 | -0.055778 | 0.183862 | 0.131931 | 1.351007 | 1.192743 | 4.443778 | 3.578594 |
| 5 | 2 | 1.249210 | 0.698349 | 0.533791 | 0.247140 | 0.472308 | 0.307934 | 1.238599 | 1.145458 | 6.332866 | 4.960891 |
|  | 3 | 0.490293 | 0.356132 | 0.166129 | 0.070080 | 0.294192 | 0.215413 | 1.232423 | 1.131607 | 5.187363 | 4.269817 |
|  | 4 | 0.290628 | 0.250441 | 0.030763 | -0.008314 | 0.202419 | 0.155572 | 1.177251 | 1.080768 | 4.291069 | 3.660222 |
|  | 5 | 0.230405 | 0.219377 | -0.029135 | -0.045832 | 0.142843 | 0.108405 | 1.084913 | 0.974634 | 3.521210 | 2.935043 |
| 6 | 2 | 1.481024 | 0.816546 | 0.631490 | 0.314617 | 0.477331 | 0.312700 | 1.056183 | 0.984097 | 5.772317 | 4.539131 |
|  | 3 | 0.575395 | 0.399459 | 0.232697 | 0.121273 | 0.301733 | 0.222413 | 1.056108 | 0.979975 | 4.819592 | 3.990282 |
|  | 4 | 0.315521 | 0.256166 | 0.080351 | 0.032039 | 0.212423 | 0.165465 | 1.029624 | 0.954972 | 4.071425 | 3.484925 |
|  | 5 | 0.223513 | 0.201595 | 0.008880 | -0.014225 | 0.155905 | 0.126963 | 0.978138 | 0.908072 | 3.462147 | 3.017064 |
|  | 6 | 0.190304 | 0.183111 | -0.027716 | -0.038896 | 0.116577 | 0.092009 | 0.905408 | 0.824072 | 2.912721 | 2.487806 |
| 7 | 2 | 1.704680 | 0.936056 | 0.713666 | 0.371176 | 0.480823 | 0.315950 | 0.927623 | 0.865509 | 5.313623 | 4.207061 |
|  | 3 | 0.667587 | 0.452368 | 0.288854 | 0.164384 | 0.306813 | 0.227011 | 0.921279 | 0.862136 | 4.592628 | 3.743848 |
|  | 4 | 0.353402 | 0.275604 | 0.122608 | 0.066457 | 0.218847 | 0.172496 | 0.909847 | 0.850044 | 3.856465 | 3.314435 |
|  | 5 | 0.233167 | 0.201413 | 0.042126 | 0.013809 | 0.164973 | 0.134936 | 0.879020 | 0.822600 | 3.336658 | 2.929802 |
|  | 6 | 0.182699 | 0.169194 | -0.001301 | -0.016396 | 0.127606 | 0.106382 | 0.834770 | 0.781725 | 2.894988 | 2.571237 |
|  | 7 | 0.162191 | 0.157140 | -0.025789 | -0.033783 | 0.098365 | 0.079907 | 0.776397 | 0.713852 | 2.480999 | 2.158902 |
| 8 | 2 | 1.918615 | 1.053689 | 0.784533 | 0.419816 | 0.483377 | 0.318312 | 0.835533 | 0.776196 | 4.929591 | 3.927166 |
|  | 3 | 0.761587 | 0.509725 | 0.337341 | 0.201542 | 0.310477 | 0.230278 | 0.816007 | 0.768891 | 4.228463 | 3.528477 |
|  | 4 | 0.398056 | 0.302514 | 0.159281 | 0.096310 | 0.223358 | 0.176631 | 0.812293 | 0.763536 | 3.659187 | 3.154194 |
|  | 5 | 0.251921 | 0.210552 | 0.071292 | 0.038447 | 0.170378 | 0.140089 | 0.793878 | 0.746945 | 3.201659 | 2.821310 |
|  | 6 | 0.185998 | 0.166819 | 0.022472 | 0.004152 | 0.134224 | 0.113074 | 0.764590 | 0.720759 | 2.819615 | 2.521431 |
|  | 7 | 0.155051 | 0.146009 | -0.005413 | -0.016979 | 0.107264 | 0.091458 | 0.727116 | 0.685586 | 2.483909 | 2.237606 |
|  | 8 | 0.141360 | 0.137624 | -0.023866 | -0.029859 | 0.085017 | 0.070624 | 0.679311 | 0.629661 | 2.160004 | 1.906837 |
| 9 | 2 | 2.122722 | 1.168113 | 0.846804 | 0.462458 | 0.485330 | 0.320106 | 0.769266 | 0.707781 | 4.602115 | 3.687256 |

Table D.I (continued)

| $n$ | $r$ | $\tilde{A}$ | $\tilde{A}^0$ | $\tilde{B}$ | $\tilde{B}^0$ | $\tilde{C}$ | $\tilde{C}^0$ | $\tilde{Q}_{.90}$ | $_{.90}$ | $\tilde{Q}_{.99}$ | $\tilde{Q}^0_{.99}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 3 | 0.856217 | 0.568920 | 0.379959 | 0.234151 | 0.313246 | 0.232724 | 0.732451 | 0.693619 | 3.989204 | 3.339422 |
| | 4 | 0.446256 | 0.333664 | 0.191609 | 0.122602 | 0.226713 | 0.173662 | 0.731980 | 0.691703 | 3.490942 | 3.007594 |
| | 5 | 0.276050 | 0.225294 | 0.097154 | 0.060297 | 0.174294 | 0.143739 | 0.721440 | 0.681829 | 3.070514 | 2.712254 |
| | 6 | 0.195796 | 0.171019 | 0.043783 | 0.022614 | 0.138801 | 0.117496 | 0.701653 | 0.664259 | 2.730209 | 2.449346 |
| | 7 | 0.155472 | 0.142875 | 0.011395 | -0.001303 | 0.112788 | 0.097092 | 0.675362 | 0.640431 | 2.437388 | 2.209473 |
| | 8 | 0.134968 | 0.128541 | -0.009069 | -0.016845 | 0.092364 | 0.080125 | 0.643529 | 0.610124 | 2.172946 | 1.979085 |
| | 9 | 0.125295 | 0.122421 | -0.022094 | -0.026751 | 0.074824 | 0.063273 | 0.603658 | 0.563247 | 1.911954 | 1.707493 |
| 10 | 2 | 2.317441 | 1.278815 | 0.902322 | 0.500402 | 0.486871 | 0.321517 | 0.721920 | 0.654849 | 4.318679 | 3.478710 |
| | 3 | 0.949076 | 0.628545 | 0.417951 | 0.263181 | 0.315415 | 0.234626 | 0.665298 | 0.632219 | 3.778412 | 3.172216 |
| | 4 | 0.496157 | 0.367223 | 0.220478 | 0.146098 | 0.229309 | 0.181987 | 0.665139 | 0.631462 | 3.320224 | 2.874531 |
| | 5 | 0.303445 | 0.243550 | 0.120331 | 0.079870 | 0.177275 | 0.146480 | 0.659619 | 0.625873 | 2.947760 | 2.608416 |
| | 6 | 0.209738 | 0.179442 | 0.062998 | 0.039269 | 0.142198 | 0.120708 | 0.646313 | 0.613982 | 2.639246 | 2.372490 |
| | 7 | 0.160661 | 0.144537 | 0.027627 | 0.013021 | 0.116706 | 0.100917 | 0.627333 | 0.596994 | 2.376135 | 2.160288 |
| | 8 | 0.134036 | 0.125239 | 0.004750 | -0.004510 | 0.097048 | 0.084935 | 0.604125 | 0.575665 | 2.144009 | 1.964086 |
| | 9 | 0.119657 | 0.114879 | -0.010439 | -0.016385 | 0.081004 | 0.071240 | 0.576856 | 0.549392 | 1.929854 | 1.773149 |
| | 10 | 0.112522 | 0.110244 | -0.020509 | -0.024229 | 0.066793 | 0.057309 | 0.543073 | 0.509513 | 1.714628 | 1.545892 |
| 11 | 2 | 2.503400 | 1.385672 | 0.952399 | 0.534571 | 0.488120 | 0.322655 | 0.688620 | 0.613683 | 4.070337 | 3.295284 |
| | 3 | 1.039956 | 0.687812 | 0.452207 | 0.289327 | 0.317159 | 0.236149 | 0.610833 | 0.581526 | 3.591040 | 3.023166 |
| | 4 | 0.546820 | 0.402057 | 0.245536 | 0.167212 | 0.231380 | 0.183828 | 0.608972 | 0.580457 | 3.174938 | 2.753762 |
| | 5 | 0.332828 | 0.264051 | 0.141299 | 0.097568 | 0.179627 | 0.148622 | 0.606536 | 0.577570 | 2.833985 | 2.511448 |
| | 6 | 0.226409 | 0.190689 | 0.080450 | 0.054395 | 0.144834 | 0.123168 | 0.597787 | 0.569612 | 2.551136 | 2.296637 |
| | 7 | 0.169057 | 0.149433 | 0.042460 | 0.026121 | 0.119670 | 0.103752 | 0.583381 | 0.557288 | 2.319788 | 2.104655 |
| | 8 | 0.136694 | 0.125541 | 0.017512 | 0.006912 | 0.100438 | 0.088273 | 0.566508 | 0.541461 | 2.100976 | 1.929932 |
| | 9 | 0.118094 | 0.111660 | 0.000584 | -0.006435 | 0.085031 | 0.075398 | 0.546077 | 0.522451 | 1.912100 | 1.766389 |
| | 10 | 0.107564 | 0.103886 | -0.011097 | -0.015782 | 0.072072 | 0.064095 | 0.522494 | 0.499503 | 1.734804 | 1.605419 |
| | 11 | 0.102120 | 0.100270 | -0.019102 | -0.022142 | 0.060304 | 0.052372 | 0.493479 | 0.465144 | 1.553971 | 1.412241 |
| 12 | 2 | 2.681270 | 1.488744 | 0.997998 | 0.565642 | 0.489152 | 0.323593 | 0.666682 | 0.581665 | 3.850507 | 3.132344 |
| | 3 | 1.128571 | 0.746274 | 0.483387 | 0.313099 | 0.318594 | 0.237396 | 0.566382 | 0.539313 | 3.423147 | 2.989310 |
| | 4 | 0.597480 | 0.437605 | 0.270268 | 0.186462 | 0.233072 | 0.185325 | 0.561389 | 0.536906 | 3.043061 | 2.643942 |
| | 5 | 0.363389 | 0.285989 | 0.160426 | 0.113702 | 0.181531 | 0.150346 | 0.560657 | 0.535624 | 2.728878 | 2.421433 |
| | 6 | 0.244901 | 0.203866 | 0.096410 | 0.068224 | 0.146945 | 0.125121 | 0.555138 | 0.530442 | 2.467467 | 2.220921 |
| | 7 | 0.179679 | 0.156593 | 0.056079 | 0.038150 | 0.122006 | 0.105959 | 0.545139 | 0.521483 | 2.245549 | 2.047843 |
| | 8 | 0.141866 | 0.128363 | 0.029301 | 0.017474 | 0.103043 | 0.090789 | 0.531815 | 0.509488 | 2.052822 | 1.888820 |
| | 9 | 0.119300 | 0.111217 | 0.010873 | 0.002876 | 0.087994 | 0.078335 | 0.515978 | 0.494975 | 1.881336 | 1.742441 |
| | 10 | 0.105732 | 0.100852 | -0.002108 | -0.007594 | 0.075575 | 0.067727 | 0.497941 | 0.478014 | 1.724395 | 1.603928 |
| | 11 | 0.097752 | 0.094840 | -0.011349 | -0.015130 | 0.064873 | 0.058229 | 0.477356 | 0.457815 | 1.574960 | 1.466240 |
| | 12 | 0.093484 | 0.091952 | -0.017855 | -0.020386 | 0.054954 | 0.049218 | 0.452143 | 0.427886 | 1.420669 | 1.299863 |
| 13 | 2 | 2.851697 | 1.588181 | 1.039851 | 0.594124 | 0.490018 | 0.324379 | 0.653132 | 0.556893 | 3.654720 | 2.986373 |
| | 3 | 1.214809 | 0.803682 | 0.511988 | 0.334886 | 0.319794 | 0.238437 | 0.529969 | 0.503931 | 3.271636 | 2.768285 |
| | 4 | 0.647783 | 0.473311 | 0.292046 | 0.204114 | 0.234481 | 0.186567 | 0.520808 | 0.499452 | 2.922805 | 2.543413 |

Table D.I (continued)

| n | r | $\tilde{A}$ | $\tilde{A}^0$ | $\tilde{B}$ | $\tilde{B}^0$ | $\tilde{C}$ | $\tilde{C}^0$ | $\tilde{Q}_{.90}$ | $\tilde{Q}^0_{.90}$ | $\tilde{Q}_{.99}$ | $\tilde{Q}^0_{.99}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 5 | 0.394596 | 0.308826 | 0.177997 | 0.128515 | 0.183107 | 0.151765 | 0.520760 | 0.493979 | 2.631767 | 2.338017 |
| | 6 | 0.264610 | 0.216375 | 0.111099 | 0.080946 | 0.148678 | 0.124713 | 0.517508 | 0.495757 | 2.388688 | 2.155082 |
| | 7 | 0.191873 | 0.165371 | 0.068647 | 0.049250 | 0.123901 | 0.107735 | 0.510365 | 0.489755 | 2.187220 | 1.997073 |
| | 8 | 0.148850 | 0.133011 | 0.040225 | 0.027262 | 0.105174 | 0.092774 | 0.500174 | 0.480133 | 2.003340 | 1.845417 |
| | 9 | 0.127503 | 0.112771 | 0.020463 | 0.011565 | 0.090302 | 0.080581 | 0.487707 | 0.468795 | 1.845150 | 1.711574 |
| | 10 | 0.106078 | 0.100000 | 0.006357 | 0.000136 | 0.078188 | 0.070333 | 0.473422 | 0.455568 | 1.702161 | 1.587102 |
| | 11 | 0.095836 | 0.092025 | -0.003882 | -0.008278 | 0.067951 | 0.061433 | 0.457424 | 0.440387 | 1.569496 | 1.468187 |
| | 12 | 0.085619 | 0.087262 | -0.011361 | -0.014474 | 0.058952 | 0.053330 | 0.439298 | 0.422477 | 1.441661 | 1.348963 |
| | 13 | 0.066197 | 0.084908 | -0.016749 | -0.018889 | 0.050470 | 0.044674 | 0.417168 | 0.396155 | 1.308307 | 1.204054 |
| 14 | 2 | 3.015280 | 1.684168 | 1.078521 | 0.620412 | 0.490757 | 0.325048 | 0.646410 | 0.537950 | 3.477651 | 2.854650 |
| | 3 | 1.298658 | 0.859900 | 0.538401 | 0.354989 | 0.320813 | 0.239318 | 0.500102 | 0.474133 | 3.134045 | 2.658196 |
| | 4 | 0.697482 | 0.508933 | 0.312161 | 0.220407 | 0.235672 | 0.197613 | 0.496007 | 0.467042 | 2.817646 | 2.451277 |
| | 5 | 0.426096 | 0.332201 | 0.194239 | 0.142200 | 0.184433 | 0.152955 | 0.485874 | 0.466785 | 2.541892 | 2.260660 |
| | 6 | 0.285120 | 0.233805 | 0.124694 | 0.092716 | 0.150126 | 0.128039 | 0.484164 | 0.464922 | 2.314764 | 2.090263 |
| | 7 | 0.205184 | 0.175323 | 0.080303 | 0.059542 | 0.125473 | 0.109198 | 0.479179 | 0.460339 | 2.121560 | 1.938311 |
| | 8 | 0.157165 | 0.139010 | 0.050382 | 0.036365 | 0.106830 | 0.094389 | 0.471412 | 0.455341 | 1.954311 | 1.801842 |
| | 9 | 0.127191 | 0.115813 | 0.029417 | 0.019680 | 0.092166 | 0.082373 | 0.461534 | 0.444389 | 1.806897 | 1.677880 |
| | 10 | 0.108035 | 0.100758 | 0.014307 | 0.007404 | 0.080248 | 0.072349 | 0.450028 | 0.433822 | 1.674554 | 1.563651 |
| | 11 | 0.095665 | 0.090958 | 0.003201 | -0.001762 | 0.070275 | 0.0637[illegible]1 | 0.437146 | 0.421794 | 1.553345 | 1.456445 |
| | 12 | 0.087717 | 0.084669 | -0.005064 | -0.008659 | 0.061682 | 0.056179 | 0.422876 | 0.408139 | 1.439585 | 1.353162 |
| | 13 | 0.082762 | 0.080817 | -0.011233 | -0.013838 | 0.054001 | 0.049179 | 0.406790 | 0.392149 | 1.328853 | 1.248834 |
| | 14 | 0.079967 | 0.078867 | -0.015764 | -0.017595 | 0.046657 | 0.041616 | 0.387194 | 0.368807 | 1.212327 | 1.121401 |
| 15 | 2 | 3.172565 | 1.776903 | 1.114456 | 0.644817 | 0.491393 | 0.325623 | 0.645184 | 0.523758 | 3.317793 | 2.735030 |
| | 3 | 1.380159 | 0.914668 | 0.562932 | 0.373646 | 0.321689 | 0.240074 | 0.475634 | 0.448958 | 3.008398 | 2.557510 |
| | 4 | 0.746429 | 0.5[illegible]4285 | 0.330844 | 0.235531 | 0.236692 | 0.198508 | 0.456035 | 0.438853 | 2.711305 | 2.366408 |
| | 5 | 0.457646 | 0.355861 | 0.209333 | 0.154911 | 0.185564 | 0.153967 | 0.455220 | 0.438360 | 2.458518 | 2.188788 |
| | 6 | 0.306140 | 0.249862 | 0.137341 | 0.103661 | 0.151356 | 0.129160 | 0.454492 | 0.437357 | 2.245454 | 2.029348 |
| | 7 | 0.219292 | 0.186129 | 0.091160 | 0.069126 | 0.126799 | 0.110428 | 0.451135 | 0.434233 | 2.063840 | 1.896950 |
| | 8 | 0.166466 | 0.146023 | 0.059864 | 0.044861 | 0.108259 | 0.095733 | 0.445271 | 0.428919 | 1.906601 | 1.759124 |
| | 9 | 0.133001 | 0.119985 | 0.037798 | 0.027277 | 0.093708 | 0.083845 | 0.437435 | 0.421823 | 1.768246 | 1.643310 |
| | 10 | 0.111219 | 0.102740 | 0.021778 | 0.014237 | 0.081926 | 0.073974 | 0.428088 | 0.413279 | 1.644525 | 1.537151 |
| | 11 | 0.096811 | 0.091205 | 0.009895 | 0.004404 | 0.072125 | 0.065581 | 0.417529 | 0.403496 | 1.532039 | 1.438475 |
| | 12 | 0.087233 | 0.083498 | 0.000946 | -0.003094 | 0.063764 | 0.058273 | 0.405886 | 0.392530 | 1.427865 | 1.345114 |
| | 13 | 0.080922 | 0.078435 | -0.005852 | -0.008842 | 0.056441 | 0.051732 | 0.393087 | 0.380210 | 1.329130 | 1.254505 |
| | 14 | 0.076897 | 0.075267 | -0.011021 | -0.013232 | 0.049802 | 0.045619 | 0.378709 | 0.365845 | 1.232185 | 1.162377 |
| | 15 | 0.074578 | 0.073629 | -0.014882 | -0.016467 | 0.043376 | 0.039350 | 0.361223 | 0.344992 | 1.129400 | 1.049367 |
| 16 | 2 | 3.374042 | 1.866582 | 1.148015 | 0.667589 | 0.491948 | 0.326124 | 0.648429 | 0.513482 | 3.172252 | 2.525796 |
| | 3 | 1.459384 | 0.968564 | 0.585828 | 0.391049 | 0.322450 | 0.240730 | 0.455667 | 0.427651 | 2.893086 | 2.464975 |
| | 4 | 0.794533 | 0.579246 | 0.348282 | 0.249640 | 0.237577 | 0.189281 | 0.430136 | 0.414232 | 2.617694 | 2.287932 |
| | 5 | 0.489080 | 0.379630 | 0.223426 | 0.166774 | 0.186542 | 0.154939 | 0.428174 | 0.413151 | 2.380970 | 2.121862 |
| | 6 | 0.327462 | 0.266338 | 0.149158 | 0.113883 | 0.152413 | 0.130121 | 0.427986 | 0.412732 | 2.180440 | 1.972116 |

Table D.I (continued)

| n | r | $\tilde{A}$ | $\tilde{A}^0$ | $\tilde{B}$ | $\tilde{B}^0$ | $\tilde{C}$ | $\tilde{C}^0$ | $\tilde{Q}_{.90}$ | $\tilde{Q}^0_{.90}$ | $\tilde{Q}_{.99}$ | $\tilde{Q}^0_{.99}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 7 | 0.233962 | 0.197559 | 0.101317 | 0.078089 | 0.127935 | 0.111476 | 0.425841 | 0.410632 | 2.009087 | 1.838101 |
| | 8 | 0.176502 | 0.153801 | 0.063748 | 0.052820 | 0.109474 | 0.096871 | 0.421479 | 0.406641 | 1.860618 | 1.717763 |
| | 9 | 0.139668 | 0.125027 | 0.045666 | 0.034409 | 0.095010 | 0.085081 | 0.41526[illegible] | 0.401021 | 1.730071 | 1.608874 |
| | 10 | 0.115350 | 0.105673 | 0.028811 | 0.020672 | 0.083327 | 0.075321 | 0.40766[illegible] | 0.394070 | 1.613600 | 1.509374 |
| | 11 | 0.098579 | 0.092471 | 0.016221 | 0.010235 | 0.073645 | 0.067061 | 0.398923 | 0.386015 | 1.508172 | 1.417414 |
| | 12 | 0.087840 | 0.083414 | 0.006658 | 0.002200 | 0.065435 | 0.059925 | 0.389248 | 0.376982 | 1.411292 | 1.331270 |
| | 13 | 0.080253 | 0.077227 | -0.000690 | -0.004038 | 0.058318 | 0.053626 | 0.379691 | 0.366971 | 1.320694 | 1.249183 |
| | 14 | 0.075144 | 0.073080 | -0.005373 | -0.008896 | 0.051997 | 0.047921 | 0.367148 | 0.355797 | 1.234107 | 1.168994 |
| | 15 | 0.071822 | 0.070436 | -0.010763 | -0.012662 | 0.046198 | 0.042533 | 0.354214 | 0.342818 | 1.148451 | 1.086989 |
| | 16 | 0.069870 | 0.069043 | -0.014090 | -0.015476 | 0.040524 | 0.036605 | 0.339505 | 0.324067 | 1.057042 | 0.986030 |
| 17 | 2 | 3.470154 | 1.953397 | 1.179492 | 0.688931 | 0.492435 | 0.326564 | 0.555343 | 0.506468 | 3.039086 | 2.525559 |
| | 3 | 1.536424 | 1.020993 | 0.607291 | 0.407353 | 0.323118 | 0.241304 | 0.439488 | 0.405612 | 2.786790 | 2.379560 |
| | 4 | 0.841748 | 0.613738 | 0.364627 | 0.262858 | 0.238351 | 0.183956 | 0.407704 | 0.392652 | 2.530904 | 2.215105 |
| | 5 | 0.520284 | 0.403381 | 0.236640 | 0.177893 | 0.187394 | 0.155598 | 0.404274 | 0.390707 | 2.308645 | 2.059385 |
| | 6 | 0.348935 | 0.283079 | 0.160244 | 0.123470 | 0.153333 | 0.130955 | 0.404222 | 0.390549 | 2.119384 | 1.918302 |
| | 7 | 0.249022 | 0.209440 | 0.110854 | 0.086502 | 0.128918 | 0.112381 | 0.402959 | 0.389232 | 1.957215 | 1.791735 |
| | 8 | 0.187087 | 0.162160 | 0.077098 | 0.060299 | 0.110571 | 0.097848 | 0.399782 | 0.386266 | 1.816533 | 1.677988 |
| | 9 | 0.146994 | 0.130745 | 0.053074 | 0.041123 | 0.096125 | 0.086134 | 0.394914 | 0.381855 | 1.692837 | 1.575110 |
| | 10 | 0.120222 | 0.109354 | 0.035446 | 0.026742 | 0.084518 | 0.076458 | 0.388700 | 0.376189 | 1.592620 | 1.491276 |
| | 11 | 0.101951 | 0.094542 | 0.022205 | 0.015752 | 0.074923 | 0.068296 | 0.381431 | 0.369509 | 1.483124 | 1.394857 |
| | 12 | 0.089306 | 0.084187 | 0.012092 | 0.007231 | 0.066819 | 0.061278 | 0.373307 | 0.361964 | 1.392120 | 1.314391 |
| | 13 | 0.080496 | 0.076929 | 0.004239 | 0.000556 | 0.059836 | 0.055132 | 0.364436 | 0.353625 | 1.307710 | 1.238492 |
| | 14 | 0.074368 | 0.071875 | -0.001893 | -0.004707 | 0.053700 | 0.049644 | 0.354631 | 0.344462 | 1.228141 | 1.165707 |
| | 15 | 0.070165 | 0.068427 | -0.005708 | -0.008864 | 0.048184 | 0.044620 | 0.344368 | 0.334283 | 1.151527 | 1.094194 |
| | 16 | 0.067383 | 0.066192 | -0.010479 | -0.012128 | 0.043071 | 0.039833 | 0.332655 | 0.322495 | 1.075236 | 1.020686 |
| | 17 | 0.065722 | 0.064995 | -0.013375 | -0.014597 | 0.038021 | 0.034526 | 0.318466 | 0.305536 | 0.993358 | 0.929904 |
| 18 | 2 | 3.611296 | 2.037522 | 1.209127 | 0.709011 | 0.492867 | 0.326953 | 0.665289 | 0.502189 | 2.916709 | 2.433176 |
| | 3 | 1.611373 | 1.072193 | 0.627488 | 0.422688 | 0.323709 | 0.241812 | 0.426524 | 0.394359 | 2.688413 | 2.300406 |
| | 4 | 0.888051 | 0.647711 | 0.383007 | 0.275290 | 0.239034 | 0.199552 | 0.388245 | 0.373687 | 2.450161 | 2.147296 |
| | 5 | 0.551180 | 0.427027 | 0.249075 | 0.188352 | 0.188145 | 0.156265 | 0.382957 | 0.370656 | 2.241014 | 2.000918 |
| | 6 | 0.370449 | 0.299972 | 0.170682 | 0.132493 | 0.154141 | 0.131685 | 0.382849 | 0.370531 | 2.061958 | 1.867639 |
| | 7 | 0.264342 | 0.221643 | 0.119838 | 0.094426 | 0.129778 | 0.113171 | 0.392199 | 0.369772 | 1.908077 | 1.747755 |
| | 8 | 0.158079 | 0.170961 | 0.084973 | 0.067351 | 0.111433 | 0.098697 | 0.379953 | 0.3676[illegible]9 | 1.774384 | 1.639881 |
| | 9 | 0.154829 | 0.136990 | 0.060068 | 0.047462 | 0.097092 | 0.087044 | 0.376167 | 0.364183 | 1.656785 | 1.542305 |
| | 10 | 0.125678 | 0.113676 | 0.041720 | 0.032483 | 0.085544 | 0.077434 | 0.371113 | 0.359570 | 1.552062 | 1.453395 |
| | 11 | 0.105764 | 0.097257 | 0.027976 | 0.020980 | 0.076015 | 0.069346 | 0.365054 | 0.354009 | 1.457684 | 1.371693 |
| | 12 | 0.091459 | 0.085645 | 0.017236 | 0.012013 | 0.067989 | 0.062415 | 0.358191 | 0.347658 | 1.371623 | 1.295909 |
| | 13 | 0.081467 | 0.077356 | 0.008940 | 0.004940 | 0.061101 | 0.056375 | 0.350656 | 0.340613 | 1.292700 | 1.224875 |
| | 14 | 0.074363 | 0.071638 | 0.002403 | -0.000685 | 0.055096 | 0.051023 | 0.342510 | 0.332939 | 1.217942 | 1.157459 |
| | 15 | 0.069333 | 0.067247 | -0.002784 | -0.005180 | 0.049736 | 0.046194 | 0.333737 | 0.324497 | 1.147449 | 1.092443 |
| | 16 | 0.065825 | 0.064345 | -0.006912 | -0.008774 | 0.044879 | 0.041735 | 0.324208 | 0.315185 | 1.079117 | 1.020233 |
| | 17 | 0.063468 | 0.062434 | -0.010185 | -0.011629 | 0.040334 | 0.037451 | 0.313564 | 0.304428 | 1.010489 | 0.961930 |

Table Γ I (continued)

| n | r | $\tilde{A}$ | $\tilde{A}^0$ | $\tilde{B}$ | $\tilde{B}^0$ | $\tilde{C}$ | $\tilde{C}^0$ | $\tilde{Q}_{.90}$ | $\tilde{Q}^0_{.90}$ | $\tilde{Q}_{.99}$ | $\tilde{Q}^0_{.99}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 18 | 0.062040 | 0.061395 | -0.012727 | -0.013813 | 0.035808 | 0.032670 | 0.300660 | 0.289009 | 0.936881 | 0.879824 |
| 19 | 2 | 3.747823 | 2.119125 | 1.237124 | 0.727969 | 0.493252 | 0.327300 | 0.677759 | 0.500225 | 2.803803 | 2.347701 |
| | 3 | 1.684328 | 1.122183 | 0.645559 | 0.437160 | 0.324235 | 0.242263 | 0.416309 | 0.381498 | 2.597038 | 2.226795 |
| | 4 | 0.933439 | 0.681137 | 0.394527 | 0.287022 | 0.239641 | 0.191080 | 0.371356 | 0.356595 | 2.374807 | 2.083964 |
| | 5 | 0.581713 | 0.450502 | 0.260817 | 0.198224 | 0.188811 | 0.154856 | 0.364011 | 0.352691 | 2.177611 | 1.945069 |
| | 6 | 0.391921 | 0.316930 | 0.180540 | 0.141012 | 0.154855 | 0.132330 | 0.363571 | 0.352414 | 2.007854 | 1.819871 |
| | 7 | 0.279825 | 0.234070 | 0.128329 | 0.101913 | 0.130537 | 0.113867 | 0.363311 | 0.352079 | 1.861508 | 1.706030 |
| | 8 | 0.209369 | 0.180055 | 0.092420 | 0.074019 | 0.112236 | 0.099442 | 0.361790 | 0.350548 | 1.734142 | 1.603441 |
| | 9 | 0.163059 | 0.143649 | 0.066689 | 0.053461 | 0.097939 | 0.087840 | 0.359887 | 0.347869 | 1.622026 | 1.510600 |
| | 10 | 0.131597 | 0.119371 | 0.047667 | 0.037923 | 0.086438 | 0.079283 | 0.354796 | 0.344125 | 1.522197 | 1.426040 |
| | 11 | 0.109693 | 0.100490 | 0.033260 | 0.025944 | 0.076962 | 0.070253 | 0.349747 | 0.339493 | 1.432318 | 1.348440 |
| | 12 | 0.094167 | 0.087661 | 0.022139 | 0.016562 | 0.068995 | 0.063387 | 0.343931 | 0.334122 | 1.350523 | 1.276543 |
| | 13 | 0.083028 | 0.078372 | 0.013424 | 0.009123 | 0.062177 | 0.057424 | 0.337487 | 0.328120 | 1.275284 | 1.209619 |
| | 14 | 0.074580 | 0.071621 | 0.006515 | 0.003168 | 0.056247 | 0.052165 | 0.330501 | 0.321549 | 1.205307 | 1.146419 |
| | 15 | 0.069158 | 0.066723 | 0.000992 | -0.001630 | 0.051008 | 0.047463 | 0.323003 | 0.314421 | 1.139422 | 1.086108 |
| | 16 | 0.064570 | 0.063203 | -0.003448 | -0.005510 | 0.046301 | 0.043180 | 0.314961 | 0.306673 | 1.076475 | 1.027648 |
| | 17 | 0.062007 | 0.060731 | -0.007023 | -0.008647 | 0.041987 | 0.039192 | 0.306244 | 0.298123 | 1.015125 | 0.969646 |
| | 18 | 0.055588 | 0.059082 | -0.009889 | -0.011163 | 0.037918 | 0.035334 | 0.296518 | 0.288264 | 0.953366 | 0.909509 |
| | 19 | 0.058749 | 0.058173 | -0.012138 | -0.013108 | 0.033837 | 0.031004 | 0.284733 | 0.274179 | 0.896456 | 0.834862 |
| 20 | 2 | 3.880054 | 2.198359 | 1.263654 | 0.745922 | 0.493598 | 0.327610 | 0.692338 | 0.500230 | 2.699270 | 2.268339 |
| | 3 | 1.755385 | 1.171007 | 0.664622 | 0.450861 | 0.324706 | 0.242667 | 0.408459 | 0.370703 | 2.511890 | 2.158116 |
| | 4 | 0.977919 | 0.713999 | 0.408277 | 0.293128 | 0.240184 | 0.191552 | 0.356707 | 0.342256 | 2.304280 | 2.074644 |
| | 5 | 0.611848 | 0.473760 | 0.271937 | 0.207570 | 0.189405 | 0.157383 | 0.347111 | 0.336555 | 2.118029 | 1.894493 |
| | 6 | 0.413294 | 0.333890 | 0.187879 | 0.149080 | 0.155493 | 0.132905 | 0.346139 | 0.335971 | 1.956790 | 1.774758 |
| | 7 | 0.295395 | 0.246645 | 0.136375 | 0.109305 | 0.131212 | 0.114485 | 0.346085 | 0.335810 | 1.817336 | 1.666472 |
| | 8 | 0.220873 | 0.189479 | 0.099492 | 0.080340 | 0.112948 | 0.100102 | 0.345115 | 0.334823 | 1.695738 | 1.568626 |
| | 9 | 0.171590 | 0.150630 | 0.072972 | 0.059153 | 0.098688 | 0.088541 | 0.342932 | 0.332784 | 1.588595 | 1.480055 |
| | 10 | 0.137883 | 0.123496 | 0.053316 | 0.043091 | 0.087226 | 0.079028 | 0.339645 | 0.329765 | 1.493175 | 1.399385 |
| | 11 | 0.114237 | 0.104145 | 0.038381 | 0.030665 | 0.077792 | 0.071044 | 0.335446 | 0.325909 | 1.407307 | 1.325414 |
| | 12 | 0.097330 | 0.090131 | 0.026809 | 0.020895 | 0.069872 | 0.064230 | 0.330511 | 0.321357 | 1.329262 | 1.257095 |
| | 13 | 0.085074 | 0.079872 | 0.017704 | 0.013116 | 0.063107 | 0.058327 | 0.324979 | 0.316216 | 1.257633 | 1.193474 |
| | 14 | 0.076108 | 0.072313 | 0.010451 | 0.006856 | 0.057241 | 0.053140 | 0.318946 | 0.310563 | 1.191245 | 1.133747 |
| | 15 | 0.069519 | 0.066732 | 0.004619 | 0.001781 | 0.052079 | 0.048522 | 0.312464 | 0.304435 | 1.129077 | 1.077126 |
| | 16 | 0.064679 | 0.062625 | -0.000103 | -0.002355 | 0.047471 | 0.044351 | 0.305545 | 0.297826 | 1.070185 | 1.022826 |
| | 17 | 0.061149 | 0.059635 | -0.003943 | -0.005735 | 0.043295 | 0.040524 | 0.298147 | 0.290670 | 1.013603 | 0.969952 |
| | 18 | 0.058619 | 0.057509 | -0.007067 | -0.008495 | 0.039437 | 0.036936 | 0.290141 | 0.282790 | 0.958178 | 0.917274 |
| | 19 | 0.056874 | 0.056074 | -0.009596 | -0.010730 | 0.035771 | 0.033441 | 0.281214 | 0.273718 | 0.902127 | 0.862456 |
| | 20 | 0.055790 | 0.055273 | -0.011599 | -0.012472 | 0.032070 | 0.029500 | 0.270405 | 0.260797 | 0.841162 | 0.794273 |
| 21 | 2 | 4.008276 | 2.275365 | 1.288861 | 0.762972 | 0.493910 | 0.327991 | 0.708698 | 0.501921 | 2.692178 | 2.194419 |
| | 3 | 1.824638 | 1.218708 | 0.681776 | 0.463864 | 0.325131 | 0.243031 | 0.402656 | 0.361707 | 2.432309 | 2.093851 |
| | 4 | 1.021506 | 0.746293 | 0.421334 | 0.308659 | 0.240674 | 0.191977 | 0.344003 | 0.329254 | 2.238094 | 1.968936 |

Table D.I (continued)

| n | r | $\tilde{A}$ | $\tilde{A}^0$ | $\tilde{B}$ | $\tilde{B}^0$ | $\tilde{C}$ | $\tilde{C}^0$ | $\tilde{Q}_{.90}$ | $\tilde{Q}^0_{.90}$ | $\tilde{Q}_{.99}$ | $\tilde{Q}^0_{.99}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 5 | 0.641562 | 0.496771 | 0.282496 | 0.216442 | 0.189940 | 0.157856 | 0.332008 | 0.322030 | 2.061906 | 1.845888 |
| | 6 | 0.434521 | 0.350804 | 0.198748 | 0.156740 | 0.156064 | 0.133419 | 0.330341 | 0.321014 | 1.908510 | 1.737082 |
| | 7 | 0.310996 | 0.255309 | 0.144020 | 0.115742 | 0.131817 | 0.115037 | 0.330340 | 0.320949 | 1.775393 | 1.628787 |
| | 8 | 0.232526 | 0.199047 | 0.106195 | 0.086347 | 0.113583 | 0.100690 | 0.329776 | 0.320332 | 1.659085 | 1.535369 |
| | 9 | 0.180354 | 0.157865 | 0.078949 | 0.064565 | 0.099355 | 0.089165 | 0.328174 | 0.318812 | 1.556487 | 1.450681 |
| | 10 | 0.144461 | 0.128924 | 0.054694 | 0.048009 | 0.087925 | 0.079689 | 0.325560 | 0.316397 | 1.455071 | 1.373524 |
| | 11 | 0.119117 | 0.108144 | 0.043260 | 0.035163 | 0.078525 | 0.071742 | 0.322081 | 0.313198 | 1.382821 | 1.302801 |
| | 12 | 0.100664 | 0.092077 | 0.031265 | 0.025031 | 0.070643 | 0.064370 | 0.317896 | 0.309338 | 1.308121 | 1.237543 |
| | 13 | 0.087520 | 0.081773 | 0.021794 | 0.016932 | 0.063921 | 0.059113 | 0.313137 | 0.304922 | 1.239666 | 1.176902 |
| | 14 | 0.077660 | 0.073428 | 0.014220 | 0.010389 | 0.058103 | 0.053980 | 0.307902 | 0.300031 | 1.176371 | 1.120130 |
| | 15 | 0.070325 | 0.067184 | 0.008101 | 0.005058 | 0.052999 | 0.049425 | 0.302258 | 0.294715 | 1.117317 | 1.066547 |
| | 16 | 0.064853 | 0.062510 | 0.003119 | 0.000686 | 0.048462 | 0.045333 | 0.296236 | 0.288995 | 1.061684 | 1.015506 |
| | 17 | 0.060780 | 0.059028 | -0.000961 | -0.002913 | 0.044377 | 0.041609 | 0.281835 | 0.282854 | 1.008694 | 0.966337 |
| | 18 | 0.057771 | 0.056462 | -0.004312 | -0.005883 | 0.040644 | 0.038168 | 0.283006 | 0.276225 | 0.957529 | 0.918263 |
| | 19 | 0.055591 | 0.054618 | -0.007063 | -0.008328 | 0.037172 | 0.034920 | 0.275625 | 0.268938 | 0.907182 | 0.870187 |
| | 20 | 0.054070 | 0.053359 | -0.009311 | -0.010325 | 0.033851 | 0.031739 | 0.267400 | 0.260560 | 0.856058 | 0.819992 |
| | 21 | 0.053114 | 0.052648 | -0.011106 | -0.011875 | 0.030478 | 0.028134 | 0.257446 | 0.248660 | 0.800256 | 0.757447 |
| 22 | 2 | 4.132751 | 2.350272 | 1.312872 | 0.779203 | 0.494193 | 0.328145 | 0.725532 | 0.505062 | 2.511739 | 2.125371 |
| | 3 | 1.892175 | 1.265330 | 0.698109 | 0.476247 | 0.325516 | 0.243360 | 0.398635 | 0.354293 | 2.357731 | 2.033555 |
| | 4 | 1.064221 | 0.778020 | 0.433762 | 0.318701 | 0.241116 | 0.192361 | 0.333022 | 0.317776 | 2.175829 | 1.916493 |
| | 5 | 0.670841 | 0.519511 | 0.292547 | 0.224885 | 0.190423 | 0.159283 | 0.318495 | 0.308933 | 2.008933 | 1.799986 |
| | 6 | 0.455570 | 0.367635 | 0.207192 | 0.164031 | 0.156580 | 0.133883 | 0.315997 | 0.307377 | 1.842782 | 1.691644 |
| | 7 | 0.326583 | 0.272016 | 0.151300 | 0.122157 | 0.132361 | 0.115534 | 0.315917 | 0.307302 | 1.735519 | 1.592990 |
| | 8 | 0.244276 | 0.208746 | 0.112590 | 0.092058 | 0.114155 | 0.101218 | 0.315636 | 0.306952 | 1.624089 | 1.503593 |
| | 9 | 0.189291 | 0.165298 | 0.084646 | 0.069725 | 0.099953 | 0.089723 | 0.314501 | 0.305850 | 1.525668 | 1.422457 |
| | 10 | 0.151270 | 0.134597 | 0.063824 | 0.052700 | 0.088550 | 0.080276 | 0.312448 | 0.303940 | 1.437917 | 1.348501 |
| | 11 | 0.124272 | 0.112423 | 0.047918 | 0.039457 | 0.079179 | 0.072363 | 0.309582 | 0.301297 | 1.358956 | 1.280716 |
| | 12 | 0.104705 | 0.096133 | 0.035523 | 0.028982 | 0.071328 | 0.065625 | 0.306039 | 0.298028 | 1.287276 | 1.218208 |
| | 13 | 0.090299 | 0.084008 | 0.025707 | 0.020584 | 0.064640 | 0.059806 | 0.301942 | 0.294229 | 1.221652 | 1.160197 |
| | 14 | 0.079566 | 0.074897 | 0.017831 | 0.013775 | 0.058860 | 0.054714 | 0.297339 | 0.289982 | 1.161077 | 1.105997 |
| | 15 | 0.071503 | 0.068007 | 0.011444 | 0.008205 | 0.053601 | 0.050208 | 0.292449 | 0.285342 | 1.104705 | 1.054998 |
| | 16 | 0.065417 | 0.062783 | 0.006220 | 0.003616 | 0.049317 | 0.046175 | 0.287168 | 0.280346 | 1.051800 | 1.006637 |
| | 17 | 0.060817 | 0.058824 | 0.001920 | -0.000185 | 0.045297 | 0.042523 | 0.281564 | 0.275000 | 1.001692 | 0.960369 |
| | 18 | 0.057351 | 0.055841 | -0.001636 | -0.003344 | 0.041648 | 0.039176 | 0.275626 | 0.269283 | 0.953732 | 0.915618 |
| | 19 | 0.054763 | 0.053620 | -0.004584 | -0.005971 | 0.038291 | 0.036063 | 0.269303 | 0.263124 | 0.907217 | 0.871700 |
| | 20 | 0.052867 | 0.052007 | -0.007025 | -0.008152 | 0.035147 | 0.033108 | 0.262474 | 0.256365 | 0.861259 | 0.827639 |
| | 21 | 0.051532 | 0.050895 | -0.009034 | -0.009947 | 0.032123 | 0.030200 | 0.254869 | 0.248601 | 0.814419 | 0.781481 |
| | 22 | 0.050684 | 0.050261 | -0.010652 | -0.011369 | 0.029036 | 0.026890 | 0.245669 | 0.237603 | 0.763131 | 0.723886 |
| 23 | 2 | 4.253712 | 2.423202 | 1.335793 | 0.794692 | 0.494451 | 0.323377 | 0.745637 | 0.509456 | 2.427275 | 2.060704 |
| | 3 | 1.958080 | 1.310915 | 0.713693 | 0.488055 | 0.325866 | 0.243660 | 0.396172 | 0.349241 | 2.287657 | 1.976943 |
| | 4 | 1.106088 | 0.809186 | 0.445620 | 0.328269 | 0.241519 | 0.192709 | 0.323560 | 0.307645 | 2.117120 | 1.867013 |
| | 5 | 0.659679 | 0.541966 | 0.302137 | 0.232937 | 0.190862 | 0.158670 | 0.306396 | 0.297107 | 1.958830 | 1.756552 |

Table D.I (continued)

| n | r | $\tilde{A}$ | $\tilde{A}^0$ | $\tilde{B}$ | $\tilde{B}^0$ | $\tilde{C}$ | $\tilde{C}^0$ | $\tilde{Q}_{.90}$ | $\tilde{Q}^0_{.90}$ | $\tilde{Q}_{.99}$ | $\tilde{Q}^0_{.99}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 6 | 0.476419 | 0.384356 | 0.215249 | 0.170986 | 0.157047 | 0.134303 | 0.302951 | 0.294922 | 1.819400 | 1.653264 |
| | 7 | 0.342171 | 0.284731 | 0.158248 | 0.128277 | 0.132854 | 0.115983 | 0.302691 | 0.294746 | 1.697565 | 1.558902 |
| | 8 | 0.256081 | 0.213536 | 0.113695 | 0.097530 | 0.114671 | 0.101694 | 0.302579 | 0.294574 | 1.597652 | 1.473217 |
| | 9 | 0.198358 | 0.172477 | 0.090086 | 0.074651 | 0.100492 | 0.090225 | 0.301811 | 0.293805 | 1.496090 | 1.395351 |
| | 10 | 0.158261 | 0.140465 | 0.068726 | 0.057182 | 0.089113 | 0.080805 | 0.300226 | 0.292315 | 1.411714 | 1.324328 |
| | 11 | 0.129648 | 0.116933 | 0.052372 | 0.043562 | 0.079766 | 0.072919 | 0.297882 | 0.290146 | 1.335768 | 1.259222 |
| | 12 | 0.108799 | 0.099546 | 0.039599 | 0.032764 | 0.071940 | 0.066210 | 0.294892 | 0.287393 | 1.266835 | 1.199204 |
| | 13 | 0.093354 | 0.086522 | 0.029457 | 0.024082 | 0.065280 | 0.060421 | 0.291367 | 0.284115 | 1.203767 | 1.143550 |
| | 14 | 0.081768 | 0.076663 | 0.021296 | 0.017023 | 0.059532 | 0.055364 | 0.287400 | 0.280419 | 1.145620 | 1.091627 |
| | 15 | 0.072995 | 0.069144 | 0.014656 | 0.011228 | 0.054507 | 0.050896 | 0.293064 | 0.276355 | 1.091605 | 1.042877 |
| | 16 | 0.066309 | 0.063383 | 0.009206 | 0.006436 | 0.050065 | 0.046907 | 0.278409 | 0.271953 | 1.041048 | 0.996794 |
| | 17 | 0.061196 | 0.058962 | 0.004700 | 0.002549 | 0.046093 | 0.043309 | 0.273465 | 0.267265 | 0.993351 | 0.952910 |
| | 18 | 0.057287 | 0.055575 | 0.000954 | -0.000884 | 0.042505 | 0.040028 | 0.268240 | 0.262262 | 0.947960 | 0.910758 |
| | 19 | 0.054309 | 0.052997 | -0.002171 | -0.003676 | 0.039225 | 0.037003 | 0.262717 | 0.256926 | 0.904323 | 0.869839 |
| | 20 | 0.052064 | .051059 | -0.004782 | -0.006016 | 0.036187 | 0.034172 | 0.256844 | 0.251190 | 0.861832 | 0.829545 |
| | 21 | 0.050403 | 0.049639 | -0.006961 | -0.007973 | 0.033327 | 0.031472 | 0.250597 | 0.244902 | 0.819695 | 0.788986 |
| | 22 | 0.049223 | 0.048650 | -0.008768 | -0.009594 | 0.030561 | 0.028801 | 0.243451 | 0.237686 | 0.776604 | 0.746396 |
| | 23 | 0.048467 | 0.048081 | -0.010233 | -0.010887 | 0.027724 | 0.025751 | 0.234920 | 0.227488 | 0.729285 | 0.693172 |
| 24 | 2 | 4.371373 | 2.494265 | 1.357719 | 0.809501 | 0.494687 | 0.326588 | 0.765809 | 0.514936 | 2.348202 | 1.999995 |
| | 3 | 2.022432 | 1.355507 | 0.728596 | 0.499342 | 0.326187 | 0.243934 | 0.395076 | 0.343618 | 2.221695 | 1.923382 |
| | 4 | 1.147133 | 0.839800 | 0.456955 | 0.337413 | 0.241886 | 0.193028 | 0.315445 | 0.298714 | 2.051647 | 1.820229 |
| | 5 | 0.728075 | 0.564126 | 0.311304 | 0.240634 | 0.191262 | 0.159023 | 0.295559 | 0.286416 | 1.911353 | 1.715375 |
| | 6 | 0.497049 | 0.400946 | 0.222952 | 0.177634 | 0.157473 | 0.134695 | 0.291070 | 0.283528 | 1.779177 | 1.616781 |
| | 7 | 0.357586 | 0.297424 | 0.164892 | 0.134128 | 0.133303 | 0.116391 | 0.290514 | 0.283173 | 1.651394 | 1.526402 |
| | 8 | 0.267908 | 0.228382 | 0.124535 | 0.102752 | 0.115140 | 0.102126 | 0.290500 | 0.283104 | 1.558681 | 1.444158 |
| | 9 | 0.207516 | 0.180572 | 0.095292 | 0.079364 | 0.100981 | 0.090680 | 0.290014 | 0.282593 | 1.467700 | 1.369317 |
| | 10 | 0.165394 | 0.146489 | 0.073418 | 0.061472 | 0.089622 | 0.081283 | 0.288817 | 0.281452 | 1.396447 | 1.300999 |
| | 11 | 0.135205 | 0.121632 | 0.056639 | 0.047494 | 0.080296 | 0.073421 | 0.286918 | 0.279686 | 1.313280 | 1.238352 |
| | 12 | 0.113101 | 0.103174 | 0.043505 | 0.036388 | 0.072492 | 0.066735 | 0.284405 | 0.277359 | 1.246867 | 1.180607 |
| | 13 | 0.096642 | 0.089271 | 0.033054 | 0.027439 | 0.065855 | 0.060972 | 0.281377 | 0.274551 | 1.186126 | 1.127088 |
| | 14 | 0.084221 | 0.078681 | 0.024623 | 0.020142 | 0.060132 | 0.055943 | 0.277921 | 0.271332 | 1.130171 | 1.077207 |
| | 15 | 0.074752 | 0.070546 | 0.017745 | 0.014136 | 0.055137 | 0.051507 | 0.274197 | 0.267763 | 1.079260 | 1.030447 |
| | 16 | 0.067480 | 0.064261 | 0.012081 | 0.009152 | 0.050726 | 0.047553 | 0.269990 | 0.263896 | 1.029765 | 0.986347 |
| | 17 | 0.061866 | 0.059390 | 0.007382 | 0.004990 | 0.046793 | 0.043996 | 0.265605 | 0.259731 | 0.984142 | 0.944488 |
| | 18 | 0.057525 | 0.055611 | 0.003460 | 0.001497 | 0.043249 | 0.040764 | 0.260971 | 0.255311 | 0.940898 | 0.904479 |
| | 19 | 0.054171 | 0.052687 | 0.000171 | -0.001446 | 0.040024 | 0.037800 | 0.256089 | 0.250621 | 0.899565 | 0.865894 |
| | 20 | 0.051591 | 0.050441 | -0.002595 | -0.003930 | 0.037059 | 0.035051 | 0.250940 | 0.245632 | 0.859677 | 0.828321 |
| | 21 | 0.049628 | 0.048739 | -0.004923 | -0.006077 | 0.034297 | 0.032466 | 0.245470 | 0.240275 | 0.820690 | 0.791207 |
| | 22 | 0.048164 | 0.047479 | -0.006880 | -0.007792 | 0.031682 | 0.029987 | 0.239572 | 0.234409 | 0.781900 | 0.753742 |
| | 23 | 0.047114 | 0.046596 | -0.008513 | -0.009264 | 0.029142 | 0.027526 | 0.233006 | 0.227683 | 0.742112 | 0.714303 |
| | 24 | 0.046435 | 0.046083 | -0.009845 | -0.010445 | 0.026524 | 0.024705 | 0.225070 | 0.218199 | 0.698307 | 0.664959 |
| 25 | 2 | 4.485928 | 2.563561 | 1.378732 | 0.823689 | 0.494903 | 0.328783 | 0.786885 | 0.521362 | 2.274010 | 1.942874 |
| | 3 | 2.085306 | 1.399146 | 0.742872 | 0.510152 | 0.326481 | 0.244185 | 0.395186 | 0.339676 | 2.159445 | 1.872879 |

Table D.I (continued)

| r | $\tilde{A}$ | $\tilde{A}^0$ | $\tilde{B}$ | $\tilde{B}^0$ | $\tilde{C}$ | $\tilde{C}^0$ | $\tilde{Q}_{.90}$ | $\tilde{Q}^0_{.90}$ | $\tilde{Q}_{.99}$ | $\tilde{Q}^0_{.99}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 1.187380 | 0.869873 | 0.467813 | 0.346170 | 0.242223 | 0.193319 | 0.308531 | 0.290854 | 2.009130 | 1.775911 |
| 5 | 0.756029 | 0.585987 | 0.320084 | 0.248003 | 0.191628 | 0.159346 | 0.285953 | 0.275745 | 1.866284 | 1.676272 |
| 6 | 0.517448 | 0.417388 | 0.230330 | 0.184000 | 0.157863 | 0.135035 | 0.280236 | 0.273088 | 1.738946 | 1.582048 |
| 7 | 0.372954 | 0.310074 | 0.171257 | 0.139731 | 0.133712 | 0.116763 | 0.279313 | 0.272489 | 1.626878 | 1.495380 |
| 8 | 0.279733 | 0.238258 | 0.130130 | 0.107755 | 0.115568 | 0.102520 | 0.279308 | 0.272456 | 1.528084 | 1.416337 |
| 9 | 0.216735 | 0.188350 | 0.100282 | 0.083881 | 0.101427 | 0.091095 | 0.279032 | 0.272141 | 1.440440 | 1.344306 |
| 10 | 0.172636 | 0.152637 | 0.077918 | 0.065585 | 0.090085 | 0.081718 | 0.278152 | 0.271288 | 1.362093 | 1.275495 |
| 11 | 0.140907 | 0.126486 | 0.060732 | 0.051266 | 0.080777 | 0.073875 | 0.276632 | 0.269864 | 1.291497 | 1.218118 |
| 12 | 0.117577 | 0.106981 | 0.047255 | 0.039867 | 0.072991 | 0.067211 | 0.274533 | 0.267915 | 1.227409 | 1.162461 |
| 13 | 0.100124 | 0.092218 | 0.036509 | 0.030662 | 0.066374 | 0.061469 | 0.271938 | 0.265506 | 1.168805 | 1.110893 |
| 14 | 0.086887 | 0.080913 | 0.027821 | 0.023141 | 0.060673 | 0.055454 | 0.268927 | 0.262703 | 1.114846 | 1.062862 |
| 15 | 0.076736 | 0.072174 | 0.020717 | 0.016934 | 0.055701 | 0.052053 | 0.265571 | 0.259563 | 1.064835 | 1.017888 |
| 16 | 0.068889 | 0.065378 | 0.014852 | 0.011770 | 0.051317 | 0.048128 | 0.261921 | 0.256132 | 1.018180 | 0.975543 |
| 17 | 0.062786 | 0.060066 | 0.009971 | 0.007443 | 0.047413 | 0.044603 | 0.258017 | 0.252441 | 0.974379 | 0.935445 |
| 18 | 0.058023 | 0.055904 | 0.005883 | 0.003799 | 0.043905 | 0.041410 | 0.253885 | 0.248512 | 0.932981 | 0.897240 |
| 19 | 0.054300 | 0.052645 | 0.002441 | 0.000715 | 0.040722 | 0.038491 | 0.249538 | 0.244350 | 0.893581 | 0.860586 |
| 20 | 0.051397 | 0.050101 | -0.000468 | -0.001901 | 0.037808 | 0.035799 | 0.244966 | 0.239946 | 0.855761 | 0.825136 |
| 21 | 0.049145 | 0.048130 | -0.002932 | -0.004124 | 0.035113 | 0.033288 | 0.240157 | 0.235270 | 0.819158 | 0.790503 |
| 22 | 0.047417 | 0.046624 | -0.005020 | -0.006013 | 0.032589 | 0.030917 | 0.235046 | 0.230256 | 0.793231 | 0.756195 |
| 23 | 0.046117 | 0.045502 | -0.006786 | -0.007612 | 0.030189 | 0.028634 | 0.229541 | 0.224769 | 0.747395 | 0.721472 |
| 24 | 0.045179 | 0.044708 | -0.008269 | -0.008954 | 0.027847 | 0.026357 | 0.223413 | 0.218483 | 0.710527 | 0.684837 |
| 25 | 0.044554 | 0.044243 | -0.009486 | -0.010037 | 0.025424 | 0.023740 | 0.215997 | 0.209638 | 0.669832 | 0.638952 |

Table D.II - Factors for Calculating Cramér-Rao Efficiencies of BLU Estimates of a Scale Parameter and 100(1-R) Percent Points of the Extreme-Value Distribution, $R = e^{-1}, .90, .99$, When $\frac{n-r}{n}$ of the Size $n$ Ordered Sample is Censored From Above, $2 \le n \le 25$, $2 \le r \le n$

| n | r | A | $A^0$ | B | $B^0$ | C | $C^0$ | $Q_{.90}$ | $Q^0_{.90}$ | $Q_{.99}$ | $Q^0_{.99}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 0.659547 | 0.554332 | 0.064322 | -0.128511 | 0.711857 | 0.303764 | 3.975007 | 2.672044 | 15.131652 | 8.168960 |
| 3 | 2 | 0.916039 | 0.514796 | 0.468246 | 0.076874 | 0.818365 | 0.399419 | 2.352912 | 2.191524 | 13.925769 | 8.259781 |
| | 3 | 0.402864 | 0.369555 | -0.024772 | -0.085674 | 0.344712 | 0.202642 | 2.260028 | 1.781363 | 7.925348 | 5.445973 |
| 4 | 2 | 1.334019 | 0.627559 | 0.772030 | 0.233942 | 0.867022 | 0.429045 | 2.250049 | 1.747402 | 12.579493 | 7.554420 |
| | 3 | 0.433157 | 0.333677 | 0.118027 | 0.009362 | 0.392233 | 0.254980 | 1.888275 | 1.582802 | 7.647457 | 5.643279 |
| | 4 | 0.293459 | 0.277166 | -0.034690 | -0.064256 | 0.225283 | 0.151982 | 1.590458 | 1.336022 | 5.379916 | 4.084480 |
| 5 | 2 | 1.789172 | 0.786604 | 1.011559 | 0.357105 | 0.895046 | 0.444948 | 1.769062 | 1.432654 | 11.422934 | 6.936842 |
| | 3 | 0.529395 | 0.362392 | 0.235374 | 0.089321 | 0.416816 | 0.274556 | 1.580857 | 1.350778 | 7.184278 | 5.350604 |
| | 4 | 0.291814 | 0.250523 | 0.038571 | -0.009858 | 0.253791 | 0.185638 | 1.403454 | 1.234990 | 5.307521 | 4.269577 |
| | 5 | 0.231395 | 0.221733 | -0.033991 | -0.051404 | 0.166647 | 0.121585 | 1.229305 | 1.068818 | 4.070604 | 3.267584 |
| 6 | 2 | 2.244005 | 0.960564 | 1.208225 | 0.457758 | 0.913293 | 0.454968 | 1.431160 | 1.204346 | 10.454506 | 6.376806 |
| | 3 | 0.652941 | 0.418373 | 0.333249 | 0.155961 | 0.432116 | 0.285030 | 1.341378 | 1.164935 | 6.731123 | 5.036280 |
| | 4 | 0.323719 | 0.257398 | 0.102022 | 0.038437 | 0.269716 | 0.199709 | 1.230427 | 1.095762 | 5.092651 | 4.129891 |
| | 5 | 0.273607 | 0.201827 | 0.010533 | -0.016292 | 0.186107 | 0.145295 | 1.118673 | 1.010952 | 4.064971 | 3.426370 |
| | 6 | 0.191174 | 0.184777 | -0.031373 | -0.042837 | 0.131960 | 0.101321 | 1.000642 | 0.890681 | 3.272275 | 2.722987 |
| 7 | 2 | 2.685691 | 1.137462 | 1.374610 | 0.542615 | 0.926126 | 0.461881 | 1.188981 | 1.034331 | 9.636969 | 5.919293 |
| | 3 | 0.787954 | 0.487326 | 0.416705 | 0.212661 | 0.442612 | 0.293479 | 1.153933 | 1.017435 | 6.320432 | 4.745446 |
| | 4 | 0.372647 | 0.280941 | 0.156958 | 0.080311 | 0.280158 | 0.208454 | 1.084985 | 0.975126 | 4.857122 | 3.953228 |
| | 5 | 0.235293 | 0.201634 | 0.050448 | 0.015963 | 0.197566 | 0.155984 | 1.008744 | 0.919716 | 3.951929 | 3.355507 |
| | 6 | 0.187701 | 0.169495 | -0.001491 | -0.018348 | 0.146271 | 0.119046 | 0.930151 | 0.854945 | 3.221719 | 2.857493 |
| | 7 | 0.167978 | 0.158381 | -0.028603 | -0.036717 | 0.109096 | 0.086847 | 0.844142 | 0.763441 | 2.734710 | 2.333989 |
| 8 | 2 | 3.109991 | 1.312232 | 1.518578 | 0.615848 | 0.935646 | 0.466946 | 1.013527 | 0.905150 | 8.938174 | 5.527468 |
| | 3 | 0.927027 | 0.562496 | 0.489237 | 0.261937 | 0.450277 | 0.299170 | 1.005371 | 0.899081 | 5.954377 | 4.434368 |
| | 4 | 0.430722 | 0.313780 | 0.205090 | 0.116971 | 0.287595 | 0.214523 | 0.964092 | 0.873701 | 4.629738 | 3.777210 |
| | 5 | 0.258047 | 0.212271 | 0.085933 | 0.044711 | 0.205369 | 0.162912 | 0.911306 | 0.836048 | 3.813329 | 3.248352 |
| | 6 | 0.186582 | 0.166839 | 0.025955 | 0.004481 | 0.155033 | 0.127490 | 0.854874 | 0.791401 | 3.228495 | 2.821640 |
| | 7 | 0.155098 | 0.146327 | -0.007184 | -0.018688 | 0.120152 | 0.100665 | 0.795897 | 0.740220 | 2.763772 | 2.448474 |
| | 8 | 0.141983 | 0.138583 | -0.026083 | -0.032128 | 0.092916 | 0.075991 | 0.729918 | 0.668011 | 2.348191 | 2.042240 |
| 9 | 2 | 3.515995 | 1.482672 | 1.645332 | 0.680191 | 0.942991 | 0.470518 | 0.886242 | 0.805608 | 8.333433 | 5.187875 |

Table D.II (continued)

| n | r | A | $A^0$ | B | $B^0$ | C | $C^0$ | $Q_{.90}$ | $Q^0_{.90}$ | $Q_{.99}$ | $Q^0_{.99}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 3 | 1.066436 | 0.640376 | 0.553268 | 0.305172 | 0.456126 | 0.303317 | 0.886216 | 0.802898 | 5.628459 | 4.251206 |
| | 4 | 0.493734 | 0.351987 | 0.247785 | 0.149453 | 0.293180 | 0.219910 | 0.863227 | 0.788441 | 4.418130 | 3.611536 |
| | 5 | 0.287481 | 0.229540 | 0.117661 | 0.070418 | 0.211085 | 0.167868 | 0.826837 | 0.762713 | 3.671814 | 3.133981 |
| | 6 | 0.198022 | 0.171599 | 0.050839 | 0.025625 | 0.161172 | 0.133140 | 0.785409 | 0.730508 | 3.140912 | 2.753261 |
| | 7 | 0.155618 | 0.142877 | 0.012844 | -0.001443 | 0.127127 | 0.107533 | 0.741601 | 0.693937 | 2.727626 | 2.431702 |
| | 8 | 0.135059 | 0.128850 | -0.009992 | -0.018312 | 0.101763 | 0.087105 | 0.695372 | 0.652381 | 2.380427 | 2.140585 |
| | 9 | 0.125823 | 0.123185 | -0.023881 | -0.028558 | 0.080876 | 0.067547 | 0.642873 | 0.593788 | 2.056979 | 1.815374 |
| 10 | 2 | 3.904149 | 1.647877 | 1.759472 | 0.737531 | 0.948830 | 0.473876 | 0.794751 | 0.728229 | 7.804222 | 4.890250 |
| | 3 | 1.204241 | 0.719043 | 0.610517 | 0.343860 | 0.460738 | 0.305552 | 0.789717 | 0.723843 | 5.337162 | 4.042479 |
| | 4 | 0.559271 | 0.393302 | 0.286078 | 0.178553 | 0.297537 | 0.222474 | 0.778479 | 0.716327 | 4.223549 | 3.458421 |
| | 5 | 0.321045 | 0.251024 | 0.145259 | 0.093577 | 0.215474 | 0.171618 | 0.753965 | 0.698961 | 3.535140 | 3.021768 |
| | 6 | 0.214365 | 0.181196 | 0.073442 | 0.044640 | 0.165771 | 0.137278 | 0.723311 | 0.675389 | 3.046613 | 2.675302 |
| | 7 | 0.161525 | 0.144726 | 0.031277 | 0.014483 | 0.132126 | 0.112245 | 0.689857 | 0.647968 | 2.669720 | 2.395734 |
| | 8 | 0.134061 | 0.125262 | 0.005260 | -0.004929 | 0.107479 | 0.092819 | 0.654675 | 0.617495 | 2.350063 | 2.134789 |
| | 9 | 0.119776 | 0.115168 | -0.011359 | -0.017641 | 0.088144 | 0.076704 | 0.617274 | 0.583010 | 2.089530 | 1.900645 |
| | 10 | 0.112973 | 0.110866 | -0.021976 | -0.025702 | 0.071573 | 0.060793 | 0.574340 | 0.534409 | 1.829746 | 1.631792 |
| 11 | 2 | 4.275424 | 1.807564 | 1.860590 | 0.789216 | 0.953583 | 0.476353 | 0.730491 | 0.667836 | 7.336566 | 4.624826 |
| | 3 | 1.339428 | 0.797401 | 0.662244 | 0.378774 | 0.464470 | 0.309156 | 0.710991 | 0.658256 | 5.075415 | 3.854744 |
| | 4 | 0.625897 | 0.436355 | 0.320751 | 0.204873 | 0.301033 | 0.225233 | 0.706758 | 0.654886 | 4.045166 | 3.317691 |
| | 5 | 0.357166 | 0.275232 | .172238 | 0.114600 | 0.218957 | 0.174566 | 0.690803 | 0.643481 | 3.405968 | 2.914349 |
| | 6 | 0.233977 | 0.194064 | 0.094075 | 0.062036 | 0.169364 | 0.140469 | 0.668254 | 0.626213 | 2.952429 | 2.595831 |
| | 7 | 0.171105 | 0.150195 | 0.048232 | 0.029145 | 0.135937 | 0.115763 | 0.642433 | 0.605264 | 2.603978 | 2.331758 |
| | 8 | 0.137035 | 0.125594 | 0.019467 | 0.007531 | 0.111652 | 0.096820 | 0.614839 | 0.581783 | 2.320632 | 2.104685 |
| | 9 | 0.118095 | 0.111705 | 0.000638 | -0.006960 | 0.092934 | 0.081546 | 0.585851 | 0.555994 | 2.078823 | 1.901371 |
| | 10 | 0.107697 | 0.104152 | -0.011959 | -0.016863 | 0.077670 | 0.069484 | 0.554854 | 0.526862 | 1.861324 | 1.708516 |
| | 11 | 0.102509 | 0.100788 | -0.020327 | -0.023366 | 0.064174 | 0.055266 | 0.518982 | 0.485826 | 1.647530 | 1.485265 |
| 12 | 2 | 4.630970 | 1.961760 | 1.953610 | 0.836245 | 0.957528 | 0.478400 | 0.687358 | 0.620733 | 6.919783 | 4.391657 |
| | 3 | 1.471483 | 0.874822 | 0.709395 | 0.410565 | 0.467553 | 0.311297 | 0.646442 | 0.603434 | 4.838895 | 3.684979 |
| | 4 | 0.692724 | 0.480282 | 0.352403 | 0.228879 | 0.303903 | 0.227484 | 0.645664 | 0.602173 | 3.891525 | 3.188403 |
| | 5 | 0.394033 | 0.301204 | 0.196008 | 0.133821 | 0.221794 | 0.176950 | 0.635855 | 0.595013 | 3.284974 | 2.814515 |
| | 6 | 0.255797 | 0.209186 | 0.113018 | 0.077981 | 0.172258 | 0.143015 | 0.619475 | 0.582467 | 2.861218 | 2.518140 |
| | 7 | 0.183261 | 0.156221 | 0.063872 | 0.042672 | 0.138960 | 0.118517 | 0.599505 | 0.566357 | 2.536204 | 2.273617 |
| | 8 | 0.142823 | 0.128699 | 0.032668 | 0.019218 | 0.114881 | 0.099855 | 0.577570 | 0.547881 | 2.273312 | 2.064945 |
| | 9 | 0.119429 | 0.111226 | 0.011922 | 0.003121 | 0.096484 | 0.084993 | 0.554380 | 0.527601 | 2.051474 | 1.881093 |
| | 10 | 0.105737 | 0.100913 | -0.002280 | -0.008146 | 0.081754 | 0.072648 | 0.530011 | 0.505476 | 1.856731 | 1.713195 |
| | 11 | 0.097890 | 0.095083 | -0.012136 | -0.016065 | 0.069373 | 0.061829 | 0.503828 | 0.480501 | 1.677577 | 1.551276 |
| | 12 | 0.093821 | 0.092389 | -0.018894 | -0.021419 | 0.058150 | 0.050661 | 0.473337 | 0.445341 | 1.498182 | 1.361493 |
| 13 | 2 | 4.571948 | 2.110639 | 2.038996 | 0.879375 | 0.960854 | 0.480120 | 0.660883 | 0.584208 | 6.545579 | 4.180131 |
| | 3 | 1.600181 | 0.950942 | 0.752696 | 0.439734 | 0.470142 | 0.313089 | 0.593368 | 0.557343 | 4.674008 | 3.530641 |
| | 4 | 0.759199 | 0.524529 | 0.381500 | 0.250929 | 0.306303 | 0.229357 | 0.593330 | 0.556665 | 3.731066 | 3.069423 |

Table D.II (continued)

| n | r | A | $A^0$ | B | $B^0$ | C | $C^0$ | $Q_{.90}$ | $Q^0_{.90}$ | $Q_{.99}$ | $Q^0_{.99}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 5 | 0.433381 | 0.328297 | 0.217895 | 0.151509 | 0.224151 | 0.178919 | 0.587825 | 0.552472 | 3.172015 | 2.720550 |
| | 6 | 0.279108 | 0.225878 | 0.130502 | 0.092691 | 0.174643 | 0.145099 | 0.576174 | 0.543508 | 2.774142 | 2.443400 |
| | 7 | 0.197252 | 0.168090 | 0.078356 | 0.055197 | 0.141424 | 0.120743 | 0.560786 | 0.531125 | 2.469087 | 2.215352 |
| | 8 | 0.150658 | 0.133830 | 0.044950 | 0.030050 | 0.117473 | 0.102261 | 0.543253 | 0.516449 | 2.223001 | 2.021349 |
| | 9 | 0.122564 | 0.112916 | 0.022495 | 0.012578 | 0.099266 | 0.087643 | 0.524419 | 0.500144 | 2.016611 | 1.851844 |
| | 10 | 0.106122 | 0.100000 | 0.006897 | 0.000146 | 0.084820 | 0.075554 | 0.504625 | 0.482468 | 1.837595 | 1.699608 |
| | 11 | 0.095852 | 0.092098 | -0.004165 | -0.008920 | 0.072905 | 0.065454 | 0.483802 | 0.463261 | 1.676950 | 1.558335 |
| | 12 | 0.089757 | 0.087483 | -0.012073 | -0.015299 | 0.062645 | 0.056334 | 0.461341 | 0.441592 | 1.526497 | 1.420261 |
| | 13 | 0.086493 | 0.085282 | -0.017639 | -0.019771 | 0.053152 | 0.046764 | 0.435055 | 0.411084 | 1.373560 | 1.256763 |
| 14 | 2 | 5.259468 | 2.254447 | 2.117889 | 0.919194 | 0.963698 | 0.481586 | 0.647723 | 0.556225 | 6.207424 | 3.988618 |
| | 3 | 1.725456 | 1.025563 | 0.792714 | 0.466672 | 0.472348 | 0.314610 | 0.549704 | 0.518430 | 4.427795 | 3.389671 |
| | 4 | 0.824573 | 0.568731 | 0.408412 | 0.271306 | 0.308338 | 0.230941 | 0.548291 | 0.517165 | 3.592325 | 2.959642 |
| | 5 | 0.472356 | 0.356073 | 0.238164 | 0.167878 | 0.226141 | 0.180575 | 0.545653 | 0.514960 | 3.066621 | 2.632763 |
| | 6 | 0.303415 | 0.243663 | 0.146721 | 0.106330 | 0.176645 | 0.146840 | 0.537619 | 0.508717 | 2.691585 | 2.372723 |
| | 7 | 0.212558 | 0.179303 | 0.091824 | 0.066840 | 0.143475 | 0.122584 | 0.525964 | 0.499258 | 2.403887 | 2.159406 |
| | 8 | 0.160007 | 0.140471 | 0.056409 | 0.040155 | 0.119608 | 0.104227 | 0.511841 | 0.487563 | 2.172107 | 1.976614 |
| | 9 | 0.128145 | 0.116235 | 0.032403 | 0.021446 | 0.101522 | 0.089767 | 0.496431 | 0.474306 | 1.978378 | 1.819525 |
| | 10 | 0.108258 | 0.100817 | 0.015556 | 0.007981 | 0.087249 | 0.077992 | 0.480089 | 0.459858 | 1.811454 | 1.677804 |
| | 11 | 0.095676 | 0.090961 | 0.003442 | -0.001882 | 0.075597 | 0.068104 | 0.462969 | 0.444318 | 1.663538 | 1.549463 |
| | 12 | 0.087744 | 0.084747 | -0.005397 | -0.009174 | 0.065737 | 0.059523 | 0.444936 | 0.427473 | 1.528480 | 1.428746 |
| | 13 | 0.082895 | 0.081018 | -0.011874 | -0.014554 | 0.057084 | 0.051723 | 0.425420 | 0.408454 | 1.400118 | 1.309469 |
| | 14 | 0.080228 | 0.079190 | -0.015535 | -0.018359 | 0.048941 | 0.043423 | 0.402490 | 0.381721 | 1.268006 | 1.165994 |
| 15 | 2 | 5.614554 | 2.393457 | 2.191194 | 0.956168 | 0.966156 | 0.482851 | 0.645331 | 0.535229 | 5.900095 | 3.814216 |
| | 3 | 1.847337 | 1.098585 | 0.829902 | 0.491687 | 0.474250 | 0.315918 | 0.513842 | 0.485486 | 4.247769 | 3.260166 |
| | 4 | 0.889828 | 0.612647 | 0.433435 | 0.290245 | 0.310088 | 0.232297 | 0.509387 | 0.482723 | 3.463989 | 2.858044 |
| | 5 | 0.511450 | 0.384226 | 0.257028 | 0.183103 | 0.227844 | 0.181987 | 0.508472 | 0.481739 | 2.969209 | 2.550722 |
| | 6 | 0.328367 | 0.267201 | 0.161836 | 0.119035 | 0.178350 | 0.149316 | 0.503178 | 0.477551 | 2.613557 | 2.305614 |
| | 7 | 0.228809 | 0.191501 | 0.104398 | 0.077707 | 0.145212 | 0.124136 | 0.494318 | 0.470404 | 2.341208 | 2.103455 |
| | 8 | 0.170484 | 0.148248 | 0.067132 | 0.049611 | 0.121402 | 0.105868 | 0.483137 | 0.461094 | 2.121875 | 1.932125 |
| | 9 | 0.134577 | 0.120797 | 0.041706 | 0.029774 | 0.103398 | 0.091519 | 0.470490 | 0.450259 | 1.939902 | 1.783535 |
| | 10 | 0.111735 | 0.102959 | 0.023721 | 0.015375 | 0.089237 | 0.079884 | 0.456882 | 0.438305 | 1.781870 | 1.651956 |
| | 11 | 0.096917 | 0.091226 | 0.010664 | 0.004713 | 0.077731 | 0.070184 | 0.442565 | 0.425435 | 1.643707 | 1.533052 |
| | 12 | 0.087234 | 0.083508 | 0.001011 | -0.003286 | 0.068107 | 0.061879 | 0.427590 | 0.411663 | 1.519172 | 1.423192 |
| | 13 | 0.080958 | 0.078517 | -0.006202 | -0.009325 | 0.059817 | 0.054554 | 0.411796 | 0.396755 | 1.403830 | 1.318746 |
| | 14 | 0.077025 | 0.075450 | -0.011599 | -0.013865 | 0.052413 | 0.047800 | 0.394655 | 0.379918 | 1.292865 | 1.214574 |
| | 15 | 0.074809 | 0.073911 | -0.015557 | -0.017135 | 0.045343 | 0.040528 | 0.374452 | 0.356273 | 1.177461 | 1.089195 |
| 16 | 2 | 5.918144 | 2.527944 | 2.259641 | 0.990671 | 0.968302 | 0.483953 | 0.651730 | 0.520007 | 5.619375 | 3.654580 |
| | 3 | 1.965907 | 1.169967 | 0.864627 | 0.515032 | 0.475906 | 0.317054 | 0.484514 | 0.457555 | 4.031917 | 3.140921 |
| | 4 | 0.953632 | 0.656116 | 0.456809 | 0.307924 | 0.311608 | 0.233473 | 0.475685 | 0.452575 | 3.344904 | 2.763734 |
| | 5 | 0.550447 | 0.412539 | 0.274662 | 0.197328 | 0.229319 | 0.183206 | 0.475573 | 0.452202 | 2.876183 | 2.473959 |
| | 6 | 0.352711 | 0.281248 | 0.175980 | 0.130918 | 0.179820 | 0.149585 | 0.472311 | 0.449540 | 2.539892 | 2.242184 |

Table D.II (continued)

| n | r | A | $A^0$ | B | $B^0$ | C | $C^0$ | $Q_{.90}$ | $Q^0_{.90}$ | $Q_{.99}$ | $Q^0_{.99}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 7 | 0.245733 | 0.204422 | 0.116181 | 0.087886 | 0.146703 | 0.125462 | 0.465762 | 0.444227 | 2.281275 | 2.050790 |
| | 8 | 0.181809 | 0.156897 | 0.077199 | 0.058485 | 0.122932 | 0.107261 | 0.456901 | 0.435851 | 2.072956 | 1.988605 |
| | 9 | 0.141972 | 0.126321 | 0.050460 | 0.037609 | 0.104985 | 0.092992 | 0.445522 | 0.427981 | 1.899343 | 1.748154 |
| | 10 | 0.116255 | 0.106135 | 0.031430 | 0.022355 | 0.090902 | 0.081456 | 0.435159 | 0.418024 | 1.750700 | 1.624176 |
| | 11 | 0.099263 | 0.092583 | 0.017510 | 0.010970 | 0.079500 | 0.071881 | 0.423052 | 0.407228 | 1.620487 | 1.512765 |
| | 12 | 0.087888 | 0.083419 | 0.007124 | 0.002340 | 0.070017 | 0.063745 | 0.410399 | 0.395700 | 1.503992 | 1.410820 |
| | 13 | 0.080254 | 0.077244 | -0.000733 | -0.004266 | 0.061930 | 0.056665 | 0.397174 | 0.383477 | 1.397514 | 1.315609 |
| | 14 | 0.075187 | 0.073163 | -0.006722 | -0.009344 | 0.054849 | 0.050333 | 0.383207 | 0.370110 | 1.297718 | 1.224237 |
| | 15 | 0.071943 | 0.070603 | -0.011284 | -0.013224 | 0.048435 | 0.044423 | 0.368014 | 0.355085 | 1.200720 | 1.132314 |
| | 16 | 0.070077 | 0.069292 | -0.014635 | -0.016064 | 0.042235 | 0.037995 | 0.350057 | 0.334006 | 1.098942 | 1.021120 |
| 17 | 2 | 6.211086 | 2.658180 | 2.323825 | 1.023009 | 0.970192 | 0.484922 | 0.665367 | 0.509604 | 5.361800 | 3.507796 |
| | 3 | 2.081279 | 1.239711 | 0.897189 | 0.536912 | 0.477363 | 0.318051 | 0.460707 | 0.433873 | 3.928523 | 3.030365 |
| | 4 | 1.016308 | 0.699035 | 0.47877[illegible] | 0.324499 | 0.312941 | 0.234502 | 0.446433 | 0.426104 | 3.234069 | 2.675926 |
| | 5 | 0.589197 | 0.440859 | 0.291211 | 0.210673 | 0.230609 | 0.184270 | 0.446371 | 0.425848 | 2.789969 | 2.402014 |
| | 6 | 0.379264 | 0.300621 | 0.189265 | 0.142075 | 0.181102 | 0.150688 | 0.444563 | 0.424286 | 2.470345 | 2.182253 |
| | 7 | 0.263129 | 0.217870 | 0.127260 | 0.097454 | 0.147997 | 0.126610 | 0.439848 | 0.420425 | 2.224130 | 2.000500 |
| | 8 | 0.193770 | 0.166190 | 0.086678 | 0.066840 | 0.124253 | 0.108461 | 0.432893 | 0.414625 | 2.025680 | 1.846427 |
| | 9 | 0.150110 | 0.132595 | 0.058718 | 0.044999 | 0.106348 | 0.094252 | 0.424396 | 0.407373 | 1.860351 | 1.713096 |
| | 10 | 0.121594 | 0.110128 | 0.038718 | 0.028956 | 0.092320 | 0.082788 | 0.414359 | 0.399054 | 1.719002 | 1.595629 |
| | 11 | 0.102484 | 0.094808 | 0.024004 | 0.016906 | 0.080991 | 0.073302 | 0.404599 | 0.389931 | 1.595520 | 1.490440 |
| | 12 | 0.089462 | 0.084242 | 0.012947 | 0.007703 | 0.071603 | 0.065278 | 0.393798 | 0.380153 | 1.485561 | 1.394752 |
| | 13 | 0.080515 | 0.076929 | 0.004509 | 0.000589 | 0.063644 | 0.058349 | 0.382527 | 0.369771 | 1.385835 | 1.306269 |
| | 14 | 0.074372 | 0.071898 | -0.002000 | -0.004953 | 0.056747 | 0.052237 | 0.370750 | 0.358725 | 1.293618 | 1.222869 |
| | 15 | 0.070212 | 0.068510 | -0.007047 | -0.009278 | 0.050624 | 0.046704 | 0.358296 | 0.346781 | 1.206316 | 1.142183 |
| | 16 | 0.067498 | 0.066346 | -0.010951 | -0.012631 | 0.045010 | 0.041485 | 0.344720 | 0.333280 | 1.120714 | 1.060437 |
| | 17 | 0.065908 | 0.065216 | -0.013904 | -0.015119 | 0.039524 | 0.035760 | 0.328641 | 0.314358 | 1.030207 | 0.961054 |
| 18 | 2 | 6.494147 | 2.784419 | 2.384241 | 1.053435 | 0.971869 | 0.485780 | 0.685006 | 0.503252 | 5.124511 | 3.372274 |
| | 3 | 2.193580 | 1.307840 | 0.927837 | 0.557497 | 0.478653 | 0.318933 | 0.441600 | 0.413822 | 3.786148 | 2.927771 |
| | 4 | 1.077817 | 0.741336 | 0.499374 | 0.340096 | 0.314119 | 0.235409 | 0.421012 | 0.402804 | 3.130615 | 2.593937 |
| | 5 | 0.627596 | 0.469074 | 0.306798 | 0.223236 | 0.231747 | 0.185206 | 0.420381 | 0.402262 | 2.709043 | 2.334459 |
| | 6 | 0.404889 | 0.320188 | 0.201785 | 0.152586 | 0.182230 | 0.151656 | 0.419549 | 0.401449 | 2.404643 | 2.125602 |
| | 7 | 0.280845 | 0.231698 | 0.137710 | 0.106476 | 0.149132 | 0.127614 | 0.416277 | 0.398730 | 2.169714 | 1.952562 |
| | 8 | 0.206205 | 0.175993 | 0.095629 | 0.074726 | 0.125408 | 0.109505 | 0.410888 | 0.394220 | 1.980191 | 1.805768 |
| | 9 | 0.158826 | 0.139458 | 0.066527 | 0.051997 | 0.107533 | 0.095343 | 0.403965 | 0.388312 | 1.822290 | 1.678760 |
| | 10 | 0.127581 | 0.114770 | 0.045623 | 0.035209 | 0.093546 | 0.083934 | 0.395976 | 0.381356 | 1.687398 | 1.566990 |
| | 11 | 0.106405 | 0.097730 | 0.030170 | 0.022544 | 0.082769 | 0.074513 | 0.387243 | 0.373614 | 1.569762 | 1.467128 |
| | 12 | 0.091777 | 0.085799 | 0.018493 | 0.012812 | 0.072949 | 0.066570 | 0.377967 | 0.365254 | 1.465329 | 1.376630 |
| | 13 | 0.081352 | 0.077382 | 0.009522 | 0.005235 | 0.065077 | 0.059743 | 0.368259 | 0.356367 | 1.371077 | 1.293457 |
| | 14 | 0.074369 | 0.071438 | 0.002543 | -0.000722 | 0.058297 | 0.053766 | 0.358148 | 0.346968 | 1.284616 | 1.215851 |
| | 15 | 0.069341 | 0.067276 | -0.002930 | -0.005431 | 0.052340 | 0.048432 | 0.347584 | 0.336984 | 1.203876 | 1.142123 |
| | 16 | 0.065875 | 0.064425 | -0.007237 | -0.009156 | 0.046988 | 0.043552 | 0.336398 | 0.326190 | 1.126779 | 1.070292 |
| | 17 | 0.063577 | 0.062575 | -0.010613 | -0.012081 | 0.042029 | 0.038908 | 0.324183 | 0.313984 | 1.050608 | 0.997068 |

Table D.II (continued)

| n | r | A | $A^0$ | B | $B^0$ | C | $C^0$ | $Q_{.90}$ | $Q^0_{.90}$ | $Q_{.99}$ | $Q^0_{.99}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 18 | 0.062208 | 0.061592 | -0.013200 | -0.014279 | 0.037138 | 0.033774 | 0.309689 | 0.296894 | 0.969538 | 0.907662 |
| 19 | 2 | 6.768016 | 2.906904 | 2.441302 | 1.082159 | 0.973368 | 0.485546 | 0.709650 | 0.500334 | 4.905119 | 3.246692 |
| | 3 | 2.302943 | 1.374393 | 0.956781 | 0.576928 | 0.479803 | 0.319719 | 0.426525 | 0.396896 | 3.653575 | 2.832170 |
| | 4 | 1.138147 | 0.782978 | 0.518870 | 0.354821 | 0.315168 | 0.236216 | 0.398914 | 0.382257 | 3.033789 | 2.517176 |
| | 5 | 0.665572 | 0.497104 | 0.321524 | 0.235101 | 0.232758 | 0.186037 | 0.397198 | 0.381097 | 2.632928 | 2.270901 |
| | 6 | 0.430438 | 0.339847 | 0.213620 | 0.162518 | 0.183230 | 0.152512 | 0.396943 | 0.380742 | 2.342509 | 2.072005 |
| | 7 | 0.258765 | 0.245791 | 0.147536 | 0.115008 | 0.150136 | 0.123499 | 0.394786 | 0.378908 | 2.117916 | 1.906897 |
| | 8 | 0.218990 | 0.186178 | 0.104105 | 0.082192 | 0.126425 | 0.110423 | 0.392681 | 0.375454 | 1.936533 | 1.766695 |
| | 9 | 0.167589 | 0.146782 | 0.073930 | 0.058609 | 0.108573 | 0.096299 | 0.385079 | 0.370668 | 1.785360 | 1.645369 |
| | 10 | 0.134084 | 0.119932 | 0.052177 | 0.041144 | 0.094617 | 0.084932 | 0.378402 | 0.364859 | 1.656258 | 1.538662 |
| | 11 | 0.110891 | 0.101214 | 0.036033 | 0.027904 | 0.083379 | 0.075561 | 0.370961 | 0.358276 | 1.543796 | 1.443461 |
| | 12 | 0.094694 | 0.087954 | 0.023779 | 0.017693 | 0.074108 | 0.067677 | 0.362967 | 0.351094 | 1.444155 | 1.357403 |
| | 13 | 0.083220 | 0.078461 | 0.014314 | 0.009678 | 0.066300 | 0.060923 | 0.354550 | 0.343423 | 1.354523 | 1.278626 |
| | 14 | 0.075025 | 0.071632 | 0.006904 | 0.003342 | 0.059600 | 0.055039 | 0.345775 | 0.335316 | 1.272719 | 1.205585 |
| | 15 | 0.069159 | 0.066725 | 0.001045 | -0.001711 | 0.053749 | 0.049828 | 0.336648 | 0.326766 | 1.196949 | 1.136905 |
| | 16 | 0.064582 | 0.063235 | -0.003615 | -0.005759 | 0.048548 | 0.045129 | 0.327110 | 0.317693 | 1.125594 | 1.071204 |
| | 17 | 0.062058 | 0.060809 | -0.007331 | -0.008999 | 0.043827 | 0.040791 | 0.317000 | 0.307884 | 1.056949 | 1.006799 |
| | 18 | 0.060090 | 0.059212 | -0.010278 | -0.011572 | 0.039413 | 0.036628 | 0.305942 | 0.296787 | 0.988678 | 0.940788 |
| | 19 | 0.058901 | 0.058351 | -0.012563 | -0.013527 | 0.035022 | 0.031996 | 0.292800 | 0.281268 | 0.915596 | 0.859891 |
| 20 | 2 | 7.033320 | 3.025855 | 2.495356 | 1.109360 | 0.974715 | 0.487233 | 0.738490 | 0.500342 | 4.701608 | 3.129932 |
| | 3 | 2.409504 | 1.439417 | 0.984196 | 0.595327 | 0.480836 | 0.320423 | 0.414926 | 0.387678 | 3.529763 | 2.742818 |
| | 4 | 1.197301 | 0.823938 | 0.537337 | 0.368765 | 0.316109 | 0.236938 | 0.379713 | 0.364114 | 2.942937 | 2.445125 |
| | 5 | 0.703077 | 0.524893 | 0.335478 | 0.246339 | 0.233662 | 0.186778 | 0.376480 | 0.362060 | 2.561194 | 2.210987 |
| | 6 | 0.455986 | 0.355521 | 0.224839 | 0.171930 | 0.184[illegible] | 0.153276 | 0.376466 | 0.361922 | 2.283674 | 2.071240 |
| | 7 | 0.316802 | 0.260064 | 0.156972 | 0.123098 | 0.151049 | 0.129286 | 0.375148 | 0.360756 | 2.058601 | 1.863398 |
| | 8 | 0.232030 | 0.196651 | 0.112149 | 0.089276 | 0.127329 | 0.111237 | 0.372092 | 0.358163 | 1.894688 | 1.729210 |
| | 9 | 0.177498 | 0.154469 | 0.080962 | 0.064899 | 0.109494 | 0.097142 | 0.367601 | 0.354317 | 1.749656 | 1.613038 |
| | 10 | 0.140997 | 0.125512 | 0.058411 | 0.046788 | 0.095561 | 0.085809 | 0.362040 | 0.349480 | 1.625802 | 1.510884 |
| | 11 | 0.115834 | 0.105157 | 0.041618 | 0.033010 | 0.084354 | 0.076478 | 0.355703 | 0.343892 | 1.517980 | 1.419825 |
| | 12 | 0.098103 | 0.090598 | 0.028823 | 0.022330 | 0.075121 | 0.068639 | 0.348801 | 0.337694 | 1.472578 | 1.337651 |
| | 13 | 0.085409 | 0.080055 | 0.018896 | 0.013928 | 0.067358 | 0.061939 | 0.341473 | 0.331038 | 1.336949 | 1.262633 |
| | 14 | 0.076224 | 0.072363 | 0.011086 | 0.007241 | 0.060716 | 0.055124 | 0.333805 | 0.323984 | 1.259068 | 1.193367 |
| | 15 | 0.069542 | 0.066735 | 0.004873 | 0.001872 | 0.054940 | 0.050996 | 0.325834 | 0.316560 | 1.197312 | 1.128656 |
| | 16 | 0.064679 | 0.062631 | -0.000108 | -0.002444 | 0.049837 | 0.046410 | 0.317549 | 0.308748 | 1.120296 | 1.067394 |
| | 17 | 0.061165 | 0.059670 | -0.004122 | -0.005978 | 0.045254 | 0.042236 | 0.308888 | 0.300462 | 1.056722 | 1.008435 |
| | 18 | 0.058671 | 0.057584 | -0.007357 | -0.008821 | 0.041056 | 0.038352 | 0.299698 | 0.291505 | 0.995162 | 0.950323 |
| | 19 | 0.056570 | 0.056193 | -0.009952 | -0.011101 | 0.037098 | 0.034598 | 0.289632 | 0.281367 | 0.933589 | 0.890475 |
| | 20 | 0.055929 | 0.055433 | -0.011984 | -0.012851 | 0.033133 | 0.030396 | 0.277655 | 0.267204 | 0.867323 | 0.816896 |
| 21 | 2 | 7.290623 | 3.141483 | 2.546703 | 1.135190 | 0.975932 | 0.487854 | 0.770861 | 0.502858 | 4.517268 | 3.021048 |
| | 3 | 2.513392 | 1.502965 | 1.010235 | 0.612796 | 0.481769 | 0.371058 | 0.405344 | 0.370817 | 3.413820 | 2.6590[illegible] |
| | 4 | 1.255295 | 0.864207 | 0.554878 | 0.382006 | 0.316957 | 0.237588 | 0.363052 | 0.348084 | 2.857488 | 2.377335 |

Table D.II (continued)

| n | r | A | $A^0$ | B | $B^0$ | C | $C^0$ | $Q_{.90}$ | $Q^0_{.90}$ | $Q_{.99}$ | $Q^0_{.99}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 5 | 0.740078 | 0.552399 | 0.348735 | 0.257013 | 0.234476 | 0.187445 | 0.357941 | 0.344904 | 2.493460 | 2.154401 |
| | 6 | 0.481326 | 0.379154 | 0.235501 | 0.180872 | 0.184924 | 0.153960 | 0.357881 | 0.344777 | 2.227891 | 1.973094 |
| | 7 | 0.334887 | 0.274447 | 0.165837 | 0.130788 | 0.151831 | 0.129991 | 0.357168 | 0.344099 | 2.021625 | 1.821945 |
| | 8 | 0.245249 | 0.207337 | 0.119802 | 0.096014 | 0.128138 | 0.111964 | 0.354960 | 0.342203 | 1.854604 | 1.693279 |
| | 9 | 0.187274 | 0.162442 | 0.087658 | 0.070887 | 0.110315 | 0.097893 | 0.351402 | 0.339145 | 1.715216 | 1.581815 |
| | 10 | 0.148238 | 0.131428 | 0.064352 | 0.052166 | 0.096401 | 0.086588 | 0.346795 | 0.335135 | 1.596157 | 1.483796 |
| | 11 | 0.121148 | 0.109476 | 0.046946 | 0.037881 | 0.085217 | 0.077287 | 0.341408 | 0.330379 | 1.492541 | 1.396465 |
| | 12 | 0.101916 | 0.093647 | 0.033641 | 0.026770 | 0.076013 | 0.069485 | 0.335445 | 0.325042 | 1.400940 | 1.317743 |
| | 13 | 0.088028 | 0.082078 | 0.023282 | 0.017996 | 0.068286 | 0.062827 | 0.329051 | 0.319247 | 1.319848 | 1.244010 |
| | 14 | 0.077875 | 0.073542 | 0.015097 | 0.010982 | 0.061697 | 0.057060 | 0.322320 | 0.313075 | 1.244363 | 1.179969 |
| | 15 | 0.070394 | 0.067211 | 0.008554 | 0.005321 | 0.055965 | 0.051995 | 0.315307 | 0.306572 | 1.175981 | 1.118536 |
| | 16 | 0.064864 | 0.062511 | 0.003278 | 0.000719 | 0.050930 | 0.047486 | 0.308031 | 0.299750 | 1.112463 | 1.060760 |
| | 17 | 0.060780 | 0.059036 | -0.001005 | -0.003040 | 0.046438 | 0.043416 | 0.300472 | 0.292591 | 1.052713 | 1.005737 |
| | 18 | 0.057791 | 0.056498 | -0.004494 | -0.006116 | 0.042366 | 0.039682 | 0.292567 | 0.284982 | 0.995668 | 0.952498 |
| | 19 | 0.055642 | 0.054690 | -0.007336 | -0.008629 | 0.038607 | 0.036183 | 0.284171 | 0.276765 | 0.940111 | 0.899769 |
| | 20 | 0.054160 | 0.053469 | -0.009637 | -0.010663 | 0.035037 | 0.032[illegible]79 | 0.274964 | 0.267462 | 0.884247 | 0.845232 |
| | 21 | 0.053242 | 0.052794 | -0.011455 | -0.012239 | 0.031436 | 0.028949 | 0.263997 | 0.254480 | 0.823870 | 0.777996 |
| 22 | 2 | 7.540436 | 3.253976 | 2.595597 | 1.159779 | 0.977038 | 0.488417 | 0.806213 | 0.507534 | 4.335635 | 2.919230 |
| | 3 | 2.614335 | 1.565091 | 1.035026 | 0.629423 | 0.482615 | 0.321633 | 0.400392 | 0.361022 | 3.304975 | 2.580402 |
| | 4 | 1.312151 | 0.903782 | 0.571579 | 0.394608 | 0.317725 | 0.239177 | 0.348631 | 0.333910 | 2.776943 | 2.313413 |
| | 5 | 0.776555 | 0.577595 | 0.361358 | 0.267174 | 0.235213 | 0.188047 | 0.341333 | 0.329418 | 2.429394 | 2.100860 |
| | 6 | 0.506469 | 0.398701 | 0.245657 | 0.189387 | 0.185648 | 0.154578 | 0.340983 | 0.329128 | 2.174927 | 1.927372 |
| | 7 | 0.352967 | 0.238888 | 0.174382 | 0.138113 | 0.152553 | 0.130625 | 0.340674 | 0.328783 | 1.976541 | 1.782414 |
| | 8 | 0.258586 | 0.218177 | 0.127099 | 0.102437 | 0.128865 | 0.112616 | 0.330142 | 0.327443 | 1.816209 | 1.658846 |
| | 9 | 0.197252 | 0.170637 | 0.094046 | 0.076598 | 0.111053 | 0.098566 | 0.336366 | 0.325045 | 1.692037 | 1.551712 |
| | 10 | 0.155739 | 0.137616 | 0.070024 | 0.057300 | 0.097153 | 0.087283 | 0.332576 | 0.321741 | 1.567388 | 1.457474 |
| | 11 | 0.126765 | 0.114102 | 0.052038 | 0.042535 | 0.085988 | 0.078008 | 0.328010 | 0.317709 | 1.467618 | 1.373531 |
| | 12 | 0.106064 | 0.097032 | 0.038252 | 0.031018 | 0.076806 | 0.070234 | 0.322862 | 0.313107 | 1.379463 | 1.297916 |
| | 13 | 0.091005 | 0.084458 | 0.027484 | 0.021893 | 0.069107 | 0.063610 | 0.317275 | 0.308052 | 1.300542 | 1.229104 |
| | 14 | 0.079903 | 0.075097 | 0.018947 | 0.014572 | 0.062542 | 0.057881 | 0.311350 | 0.302632 | 1.229055 | 1.165879 |
| | 15 | 0.071641 | 0.068078 | 0.012095 | 0.008639 | 0.056860 | 0.052362 | 0.305151 | 0.296902 | 1.163591 | 1.107243 |
| | 16 | 0.065457 | 0.062797 | 0.006543 | 0.003791 | 0.051875 | 0.049410 | 0.298712 | 0.290891 | 1.103010 | 1.052343 |
| | 17 | 0.060820 | 0.058824 | 0.002011 | -0.000193 | 0.047446 | 0.044411 | 0.292042 | 0.284600 | 1.046337 | 1.000407 |
| | 18 | 0.057353 | 0.055853 | -0.001707 | -0.003480 | 0.043458 | 0.040773 | 0.285115 | 0.277997 | 0.992690 | 0.950685 |
| | 19 | 0.054785 | 0.053657 | -0.004766 | -0.006195 | 0.039815 | 0.037412 | 0.277866 | 0.270999 | 0.941180 | 0.902344 |
| | 20 | 0.052918 | 0.052076 | -0.007281 | -0.008431 | 0.036427 | 0.034242 | 0.270161 | 0.263431 | 0.890758 | 0.854258 |
| | 21 | 0.051616 | 0.050997 | -0.009334 | -0.010257 | 0.033189 | 0.031140 | 0.261702 | 0.254860 | 0.839823 | 0.804333 |
| | 22 | 0.050801 | 0.050394 | -0.010970 | -0.011683 | 0.029905 | 0.027633 | 0.251617 | 0.242913 | 0.784553 | 0.742633 |
| 23 | 2 | 7.783223 | 3.363513 | 2.642260 | 1.183240 | 0.978046 | 0.488930 | 0.844089 | 0.514079 | 4.170447 | 2.823779 |
| | 3 | 2.713654 | 1.625850 | 1.058682 | 0.645285 | 0.483385 | 0.322156 | 0.396743 | 0.353044 | 3.202559 | 2.506310 |
| | 4 | 1.367897 | 0.942670 | 0.587515 | 0.406630 | 0.318424 | 0.239711 | 0.336[illegible]92 | 0.321406 | 2.700864 | 2.253012 |
| | 5 | 0.812499 | 0.606459 | 0.37[illegible]405 | 0.276868 | 0.235883 | 0.188594 | 0.32[illegible]447 | 0.315421 | 2.368661 | 2.050109 |

Table D.II (continued)

| n | r | A | $A^0$ | B | $B^0$ | C | $C^0$ | $Q_{.90}$ | $Q^0_{.90}$ | $Q_{.99}$ | $Q^0_{.99}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 6 | 0.531383 | 0.418128 | 0.255351 | 0.197513 | 0.186306 | 0.155138 | 0.325597 | 0.314820 | 2.124570 | 1.883893 |
| | 7 | 0.371001 | 0.303345 | 0.182493 | 0.145107 | 0.153208 | 0.131200 | 0.325518 | 0.314674 | 1.934110 | 1.744687 |
| | 8 | 0.271994 | 0.229125 | 0.134069 | 0.108570 | 0.129574 | 0.113206 | 0.324514 | 0.313771 | 1.779425 | 1.625843 |
| | 9 | 0.207380 | 0.179003 | 0.100151 | 0.082055 | 0.111719 | 0.099173 | 0.322391 | 0.311923 | 1.650094 | 1.522713 |
| | 10 | 0.163446 | 0.144022 | 0.075449 | 0.062209 | 0.097831 | 0.087909 | 0.319299 | 0.309221 | 1.539526 | 1.431959 |
| | 11 | 0.132629 | 0.118380 | 0.056912 | 0.046989 | 0.086680 | 0.078555 | 0.315445 | 0.305817 | 1.443295 | 1.351116 |
| | 12 | 0.110488 | 0.100696 | 0.042668 | 0.035087 | 0.077517 | 0.070905 | 0.311007 | 0.301851 | 1.358292 | 1.278325 |
| | 13 | 0.094282 | 0.087139 | 0.031514 | 0.025631 | 0.069840 | 0.064307 | 0.306124 | 0.297439 | 1.282246 | 1.212141 |
| | 14 | 0.082250 | 0.076969 | 0.022644 | 0.018021 | 0.063300 | 0.058609 | 0.300899 | 0.292668 | 1.213444 | 1.151420 |
| | 15 | 0.073222 | 0.069277 | 0.015501 | 0.011831 | 0.057650 | 0.053626 | 0.295402 | 0.287599 | 1.150555 | 1.095227 |
| | 16 | 0.066398 | 0.063427 | 0.009691 | 0.006753 | 0.052703 | 0.049216 | 0.289677 | 0.282271 | 1.092509 | 1.042777 |
| | 17 | 0.061219 | 0.058968 | 0.004927 | 0.002559 | 0.048321 | 0.045269 | 0.283745 | 0.276701 | 1.038416 | 0.993385 |
| | 18 | 0.057287 | 0.055576 | 0.000997 | -0.000921 | 0.044391 | 0.041697 | 0.277606 | 0.270841 | 0.987499 | 0.946418 |
| | 19 | 0.054314 | 0.053011 | -0.002259 | -0.003817 | 0.040826 | 0.038424 | 0.271230 | 0.264776 | 0.939031 | 0.901240 |
| | 20 | 0.052088 | 0.051097 | -0.004961 | -0.006229 | 0.037546 | 0.035382 | 0.264557 | 0.258309 | 0.892260 | 0.857127 |
| | 21 | 0.050454 | 0.049704 | -0.007201 | -0.008232 | 0.034476 | 0.032495 | 0.257456 | 0.251[illegible] | 0.846267 | 0.813076 |
| | 22 | 0.049302 | 0.048745 | -0.009045 | -0.009879 | 0.031524 | 0.029656 | 0.249654 | 0.243386 | 0.799614 | 0.767183 |
| | 23 | 0.048575 | 0.048203 | -0.010525 | -0.011175 | 0.028514 | 0.026432 | 0.240344 | 0.232352 | 0.748807 | 0.710344 |
| 24 | 2 | 8.019406 | 3.470256 | 2.686884 | 1.205671 | 0.978970 | 0.489399 | 0.884105 | 0.522246 | 4.015612 | 2.734091 |
| | 3 | 2.810263 | 1.685296 | 1.081302 | 0.660447 | 0.484091 | 0.322635 | 0.395120 | 0.346672 | 3.105988 | 2.436387 |
| | 4 | 1.422564 | 0.980880 | 0.602753 | 0.418122 | 0.319063 | 0.239200 | 0.325516 | 0.310367 | 2.628856 | 2.195820 |
| | 5 | 0.847903 | 0.632980 | 0.384925 | 0.286136 | 0.236494 | 0.189093 | 0.313100 | 0.302757 | 2.311020 | 2.001923 |
| | 6 | 0.556047 | 0.437411 | 0.264623 | 0.205283 | 0.186906 | 0.155549 | 0.311570 | 0.301717 | 2.076626 | 1.842489 |
| | 7 | 0.388957 | 0.317784 | 0.190253 | 0.151795 | 0.153805 | 0.131722 | 0.311569 | 0.301656 | 1.893297 | 1.708646 |
| | 8 | 0.285435 | 0.240141 | 0.140739 | 0.114439 | 0.130123 | 0.113742 | 0.310966 | 0.301087 | 1.744167 | 1.594202 |
| | 9 | 0.217616 | 0.187499 | 0.105996 | 0.087279 | 0.112324 | 0.099723 | 0.309382 | 0.299693 | 1.619350 | 1.494789 |
| | 10 | 0.171314 | 0.150602 | 0.080646 | 0.066911 | 0.098445 | 0.088675 | 0.306888 | 0.297506 | 1.512574 | 1.407258 |
| | 11 | 0.138693 | 0.124067 | 0.061584 | 0.051258 | 0.087306 | 0.079238 | 0.303652 | 0.294644 | 1.419621 | 1.329273 |
| | 12 | 0.115142 | 0.104593 | 0.046906 | 0.038990 | 0.078158 | 0.071508 | 0.299835 | 0.291234 | 1.337520 | 1.259070 |
| | 13 | 0.097811 | 0.090073 | 0.035384 | 0.029220 | 0.070498 | 0.064731 | 0.295570 | 0.287383 | 1.264102 | 1.195277 |
| | 14 | 0.084866 | 0.079110 | 0.026198 | 0.021336 | 0.063980 | 0.059259 | 0.290959 | 0.283178 | 1.197735 | 1.136808 |
| | 15 | 0.075086 | 0.070757 | 0.018780 | 0.014904 | 0.058354 | 0.054304 | 0.286074 | 0.278683 | 1.137151 | 1.082734 |
| | 16 | 0.067633 | 0.064349 | 0.012727 | 0.009609 | 0.053437 | 0.049927 | 0.280965 | 0.273940 | 1.081340 | 1.032473 |
| | 17 | 0.061924 | 0.059416 | 0.007745 | 0.005220 | 0.049090 | 0.046020 | 0.275664 | 0.268977 | 1.029473 | 0.985247 |
| | 18 | 0.057538 | 0.055613 | 0.003617 | 0.001560 | 0.045204 | 0.042497 | 0.270180 | 0.263890 | 0.980842 | 0.940547 |
| | 19 | 0.054171 | 0.052689 | 0.000178 | -0.001503 | 0.041693 | 0.039285 | 0.264508 | 0.258399 | 0.934812 | 0.897843 |
| | 20 | 0.051598 | 0.050457 | -0.002695 | -0.004073 | 0.038485 | 0.036324 | 0.258619 | 0.252738 | 0.890783 | 0.856592 |
| | 21 | 0.049653 | 0.048776 | -0.005098 | -0.006229 | 0.035515 | 0.033555 | 0.252450 | 0.246740 | 0.848099 | 0.816158 |
| | 22 | 0.048212 | 0.047542 | -0.007105 | -0.008033 | 0.032719 | 0.030914 | 0.245883 | 0.240250 | 0.805956 | 0.775638 |
| | 23 | 0.047188 | 0.046684 | -0.008768 | -0.009526 | 0.030016 | 0.028305 | 0.239660 | 0.232897 | 0.763048 | 0.733291 |
| | 24 | 0.046535 | 0.046194 | -0.010114 | -0.010709 | 0.027247 | 0.025330 | 0.230037 | 0.222670 | 0.716169 | 0.680747 |
| 25 | 2 | 8.249372 | 3.574357 | 2.729640 | 1.227157 | 0.979819 | 0.489831 | 0.925937 | 0.531826 | 3.870176 | 2.649639 |
| | 3 | 2.904672 | 1.743482 | 1.102970 | 0.674969 | 0.484738 | 0.323074 | 0.395285 | 0.341725 | 3.014748 | 2.370267 |

Table D.II (continued)

| r | A | $A^0$ | B | $B^0$ | C | $C^0$ | $Q_{.90}$ | $Q^0_{.90}$ | $Q_{.99}$ | $Q^0_{.99}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 1.476105 | 1.018425 | 0.617349 | 0.429129 | 0.319649 | 0.239648 | 0.316412 | 0.300644 | 2.560606 | 2.141593 |
| 5 | 0.882770 | 0.659151 | 0.395961 | 0.295012 | 0.237055 | 0.189551 | 0.301136 | 0.291292 | 2.256217 | 1.956100 |
| 6 | 0.580445 | 0.456530 | 0.273506 | 0.212726 | 0.187455 | 0.156116 | 0.299768 | 0.289702 | 2.030919 | 1.803011 |
| 7 | 0.406810 | 0.332180 | 0.197690 | 0.158204 | 0.154391 | 0.132200 | 0.298715 | 0.289626 | 1.854278 | 1.674183 |
| 8 | 0.298879 | 0.251196 | 0.147134 | 0.120364 | 0.130670 | 0.114231 | 0.298399 | 0.289390 | 1.710353 | 1.563848 |
| 9 | 0.227927 | 0.196091 | 0.111602 | 0.092288 | 0.112876 | 0.100224 | 0.297256 | 0.288780 | 1.589758 | 1.467903 |
| 10 | 0.179308 | 0.157321 | 0.085632 | 0.071421 | 0.099004 | 0.088990 | 0.295271 | 0.286531 | 1.486523 | 1.383366 |
| 11 | 0.144919 | 0.129324 | 0.066069 | 0.055356 | 0.087875 | 0.079768 | 0.292577 | 0.284139 | 1.396618 | 1.308033 |
| 12 | 0.119986 | 0.108685 | 0.050976 | 0.042740 | 0.078738 | 0.072053 | 0.289300 | 0.281215 | 1.317205 | 1.240215 |
| 13 | 0.101552 | 0.093720 | 0.039104 | 0.032670 | 0.071093 | 0.065495 | 0.285580 | 0.277857 | 1.246209 | 1.178613 |
| 14 | 0.087711 | 0.081480 | 0.029618 | 0.024526 | 0.064592 | 0.059843 | 0.281510 | 0.274149 | 1.182068 | 1.122193 |
| 15 | 0.077191 | 0.072478 | 0.021939 | 0.017564 | 0.058986 | 0.054911 | 0.277164 | 0.270156 | 1.123574 | 1.070121 |
| 16 | 0.069122 | 0.065524 | 0.015455 | 0.012365 | 0.054093 | 0.050561 | 0.272595 | 0.265922 | 1.069763 | 1.021708 |
| 17 | 0.062891 | 0.060124 | 0.010467 | 0.007791 | 0.049773 | 0.046685 | 0.267839 | 0.261480 | 1.019856 | 0.976370 |
| 18 | 0.058059 | 0.055919 | 0.006153 | 0.003963 | 0.045921 | 0.043198 | 0.262916 | 0.256847 | 0.973196 | 0.933599 |
| 19 | 0.054307 | 0.052645 | 0.002545 | 0.000744 | 0.042451 | 0.040032 | 0.257832 | 0.252024 | 0.929215 | 0.892930 |
| 20 | 0.051398 | 0.050105 | -0.000486 | -0.001972 | 0.039293 | 0.037128 | 0.252572 | 0.246998 | 0.887367 | 0.853915 |
| 21 | 0.049153 | 0.048148 | -0.003038 | -0.004266 | 0.036391 | 0.034435 | 0.247117 | 0.241731 | 0.847188 | 0.816083 |
| 22 | 0.047443 | 0.046661 | -0.005189 | -0.006205 | 0.033687 | 0.031903 | 0.241393 | 0.236152 | 0.808044 | 0.778870 |
| 23 | 0.046165 | 0.045562 | -0.006997 | -0.007837 | 0.031129 | 0.029478 | 0.235299 | 0.230114 | 0.769272 | 0.741458 |
| 24 | 0.045249 | 0.044791 | -0.008505 | -0.009196 | 0.028644 | 0.027070 | 0.228589 | 0.223270 | 0.729656 | 0.702248 |
| 25 | 0.044646 | 0.044347 | -0.009733 | -0.010281 | 0.026087 | 0.024317 | 0.220562 | 0.213764 | 0.686237 | 0.653517 |

Table D.III - Cramér-Rao Efficiencies of BLI Estimates of 10 Percent Points of the Extreme-Value Distribution, Where $\frac{n-r}{n}$ of the Size n Ordered Sample is Censored From Above, $2 \le n \le 18$, $2 \le r \le n$

| n/r | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | .835 | | | | | | | | | | | | | | | | |
| 3 | .892 | .862 | | | | | | | | | | | | | | | |
| 4 | .912 | .9[illegible]5 | .883 | | | | | | | | | | | | | | |
| 5 | .925 | .918 | .918 | .898 | | | | | | | | | | | | | |
| 6 | .932 | .928 | .927 | .928 | .910 | | | | | | | | | | | | |
| 7 | .933 | .936 | .934 | .936 | .936 | .919 | | | | | | | | | | | |
| 8 | .929 | .942 | .940 | .941 | .943 | .943 | .927 | | | | | | | | | | |
| 9 | .920 | .947 | .945 | .945 | .947 | .948 | .948 | .933 | | | | | | | | | |
| 10 | .907 | .950 | .949 | .949 | .950 | .952 | .953 | .952 | .938 | | | | | | | | |
| 11 | .891 | .952 | .953 | .952 | .953 | .954 | .956 | .957 | .956 | .943 | | | | | | | |
| 12 | .872 | .952 | .956 | .955 | .956 | .957 | .958 | .959 | .960 | .959 | .946 | | | | | | |
| 13 | [illegible] | .951 | .959 | .958 | .958 | .959 | .960 | .961 | .962 | .963 | .962 | .950 | | | | | |
| 14 | .852 | .948 | .961 | .961 | .960 | .961 | .962 | .963 | .964 | .965 | .965 | .964 | .953 | | | | |
| 15 | .812 | .944 | .962 | .963 | .962 | .963 | .963 | .964 | .965 | .966 | .967 | .967 | .966 | .955 | | | |
| 16 | .792 | .939 | .963 | .965 | .964 | .964 | .965 | .966 | .967 | .968 | .968 | .969 | .969 | .968 | .957 | | |
| 17 | .773 | .932 | .963 | .967 | .966 | .966 | .966 | .967 | .968 | .969 | .970 | .970 | .971 | .971 | .969 | .959 | |
| 18 | .755 | .925 | .963 | .968 | .968 | .967 | .968 | .968 | .969 | .970 | .971 | .971 | .972 | .972 | .972 | .971 | .961 |

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13. ABSTRACT

This report gives results concerning estimation of location and scale parameters. Most of the work pertains to the first extreme-value distribution of smallest values, the distribution of the natural logarithms of failure times having the two-parameter Weibull distribution. Experimental designs are derived, under the assumption that log failure times are polynomial functions of the reciprocal of stress level and have the extreme-value distribution, for over-stress life tests. These designs yield least-squares curves with minimum variance at a specified (nominal) stress level below the levels at which the life test is conducted. An estimate of the extreme-value location parameter u associated with the nominal stress level and the relationship between u and stress level can be obtained from the least-squares curve. Other extreme-value results apply to a life test conducted at a single fixed stress level.

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